Received: *January 26, 2001,* Accepted: *February 23, 2001*

Hypermultiplets, hyper-Kähler cones and quaternion-Kähler geometry

Bernard de Wit

Institute for Theoretical Physics and Spinoza Institute
Utrecht University, Utrecht, The Netherlands
E-mail: B.deWit@phys.uu.nl

Martin Roček and Stefan Vandoren

C.N. Yang Institute for Theoretical Physics
SUNY, Stony Brook, NY 11794-3840, USA
E-mail: rocek@insti.physics.sunysb.edu,
vandoren@insti.physics.sunysb.edu

Abstract: We study hyper-Kähler cones and their corresponding quaternion-Kähler spaces. We present a classification of $4(n-1)$-dimensional quaternion-Kähler spaces with n abelian quaternionic isometries, based on dualizing superconformal tensor multiplets. These manifolds characterize the geometry of the hypermultiplet sector of perturbative moduli spaces of type-II strings compactified on a Calabi-Yau manifold. As an example of our construction, we study the universal hypermultiplet in detail, and give three inequivalent tensor multiplet descriptions. We also comment on the construction of quaternion-Kähler manifolds that may describe instanton corrections to the moduli space.

Keywords: Extended Supersymmetry, Supergravity Models.

JHEP02(2001)039

Contents

JHEP02(2001)039

1. Introduction

Hyper-Kähler and quaternion-Kähler manifolds, whose real dimensions are multiples of four, appear in various contexts in field and string theory. By definition, hyper-Kähler spaces of dimension $4n$ have a holonomy group contained in $\mathrm{Sp}(n)$; they are Ricci flat. Examples of hyper-Kähler spaces are the moduli spaces of magnetic monopoles (such as the Taub-Nut and Atiyah-Hitchin manifolds [1]), or the moduli spaces of Yang-Mills instantons in flat space, as described by the ADHM construction [2]. Other examples of hyper-Kähler spaces are four-dimensional gravitational instantons (such as the Eguchi-Hanson metric) and K3 surfaces. Furthermore, in rigidly supersymmetric sigma models with 8 supercharges the scalar fields are known to parametrize a hyper-Kähler target space[1] [3, 4]. In four spacetime dimensions such a sigma model has $N = 2$ supersymmetry and is based on n hypermultiplets, each consisting of four real scalars and two Majorana spinor fields. In what follows, we work in four spacetime dimensions, though many of our conclusions concern the geometry of the target-space, and are independent of the spacetime dimension.

When the supersymmetry is realized locally, the hypermultiplets couple to $N = 2$ supergravity and the target space becomes quaternion-Kähler [5]. Quaternion-Kähler spaces of dimension $4n$ have a holonomy group contained in $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$, with nontrivial $\mathrm{Sp}(1)$ holonomy; they are Einstein spaces. The simplest (compact) four-dimensional quaternion-Kähler spaces are the sphere S^4 and the complex projective space $CP^2 = \mathrm{SU}(3)/\mathrm{U}(2)$. Quaternion-Kähler spaces appear as (part of) the moduli space of Calabi-Yau manifolds. Therefore they appear as hypermultiplet target spaces in the low-energy effective action for type-II superstrings compactified on a Calabi-Yau manifold. Classically these moduli spaces are known [6], but little is understood about the perturbative [7] and non-perturbative [8] corrections to them. One reason is that our knowledge of quaternion-Kähler geometry is limited, and no convenient formulation is known that allows one to address these questions effectively.

From the $N = 2$ superconformal multiplet calculus [9] it is clear that there exists a relation between quaternion-Kähler manifolds and certain hyper-Kähler spaces. These are *hyper-Kähler cones*[2] and correspond to field theories that are invariant under rigid $N = 2$ superconformal symmetry [10, 11]. Such a cone (relevant, e.g. for the moduli space of Yang-Mills instantons) has a homothetic Killing vector and three complex structures which rotate isometrically under the group $\mathrm{Sp}(1)$ [12, 13]. It is a cone over a $(4n-3)$-dimensional 3-sasakian manifold, which in turn is an $\mathrm{Sp}(1)$ fibration of a $(4n-4)$-dimensional quaternion-Kähler space. The quaternion-Kähler manifold is the $N = 2$ *superconformal quotient* of the hyper-Kähler manifold. There is a one-to-one relation between quaternion-Kähler spaces and hyper-Kähler cones; it has been extended to the case with torsion in [14].

In this paper we give a detailed description of the $N = 2$ superconformal quotient and study other aspects of hyper-Kähler cones, such as their isometry structure and their dual description in terms of rigidly superconformally invariant actions of tensor multiplets. The superconformal quotient can be performed in two steps: First, one descends from the hyper-Kähler cone to the *twistor space* [15, 12], which is a

[1]To be precise, this is true only for spacetime (or worldsheet, as opposed to target-space) dimension greater than two; in two dimensions, target-space torsion can modify the geometry.

[2]In [11] these spaces were called *special* hyper-Kähler, but to avoid confusion with hyper-Kähler manifolds related by the **c**-map to *special* Kähler geometries, we have changed our nomenclature and call them hyper-Kähler cones (HKC). We thank A. Van Proeyen for his encouragement on this issue.

Kähler quotient of the hyper-Kähler cone and is an S^2 fibration of the underlying quaternion-Kähler space. The twistor space plays an intermediate role in the explicit construction of the quaternion-Kähler space. Then one projects from the twistor space down to the quaternion-Kähler space, which can be done by imposing a gauge condition. Triholomorphic isometries of a hyper-Kähler cone lead to quaternionic isometries (isometries that rotate the quaternionic structure) on the corresponding quaternion-Kähler space.

Any $N = 2$ superconformal theory of n tensor multiplets has a dual description in terms of n hypermultiplets whose target space is a hyper-Kähler cone with n abelian triholomorphic isometries. For such a space there is a systematic representation in terms of a homogeneous function of the tensor multiplet scalars and an auxiliary complex variable ξ integrated along a closed loop in the complex ξ-plane [16, 17]. For the hyper-Kähler cones, this function is remarkably simple (as compared to the functions that appear in the lagrangian of the tensor multiplets), and therefore provides an effective way of studying these spaces. Performing their $N = 2$ superconformal quotient yields quaternion-Kähler spaces of real dimension $4n - 4$ with n abelian quaternionic isometries. Precisely these manifolds occur in the low-energy effective actions of type-II superstrings on Calabi-Yau manifolds, because the perturbative corrections to the hypermultiplet moduli spaces respect certain Peccei-Quinn isometries and hence fall into this class. As all quaternion-Kähler spaces with n abelian quaternionic isometries can be constructed in terms of tensor multiplets, one obtains a general classification of these spaces in terms of the homogeneous functions mentioned above. However, the homogeneous functions fall into equivalence classes, so that different functions can lead to the same hyper-Kähler cone or corresponding quaternion-Kähler space. This aspect is studied at the end of this paper.

The paper is organized as follows. In section 2 we review aspects of hyper-Kähler cones and quaternion-Kähler geometry. This section summarizes some of the results of [11] and emphasizes the relevance of hyper-Kähler quotients and superconformal quotients. In section 3 we construct the twistor space, which is an Einstein-Kähler manifold of dimension $4n-2$ and plays an intermediate role in the description of the superconformal quotient. We also discuss the isometries of the twistor space that descend from triholomorphic isometries of the corresponding hyper-Kähler cone. In section 4 we derive the $(4n - 4)$-dimensional quaternion-Kähler geometry that corresponds to the twistor space and give explicit formulae for the quaternion-Kähler metric and quaternionic structure in terms of twistor space quantities. We also exhibit how hyper-Kähler cone and twistor space isometries descend to the quaternion-Kähler manifold. In section 5 we construct general lagrangians for n $N = 2$ tensor supermultiplets that are rigidly superconformally invariant. These actions are encoded in homogeneous functions, which, as mentioned above, have a contour integral representation. The tensor fields can be dualized so that one obtains a field theory of n hypermultiplets whose target space is a hyper-Kähler cone with n abelian triholomorphic isometries. In section 6 we study the corresponding quaternion-Kähler spaces with n abelian quaternionic isometries and explain their classification. In section 7 we discuss the geometry of unitary Wolf spaces, and specifically the universal hypermultiplet from the various points of view developed in this paper. We describe them as coset spaces, as combined hyper-Kähler and superconformal quotients, and in terms of tensor multiplets. We note the appearence of inequivalent tensor multiplet descriptions. In section 8 we discuss our results from the point of view of applications and mention open problems and future perspectives.

We have added two appendices. In appendix A we discuss and derive the restrictions on functions that encode the superconformally invariant tensor multiplet lagrangians. In appendix B we present a self-contained description of projective superspace (from which the contour integral representation arises naturally), discuss gauging triholomorphic isometries, and give applications to hyper-Kähler quotients and tensor multiplet dualities.

A brief summary of our main results will appear in [18].

2. Preliminaries

In this section, we briefly review properties of hyper-Kähler cones and discuss Kähler and superconformal quaternion-Kähler quotients.

2.1 Hyper-Kähler cones

Hyper-Kähler cones [10]–[13] have a homothetic conformal Killing vector χ^A:

$$D_A\chi^B = \delta_A{}^B\,, \qquad (A, B = 1, \ldots, 4n)\,. \tag{2.1}$$

Hence the hyper-Kähler cone can be characterized by a hyper-Kähler potential χ, which serves as a Kähler potential for each of the three complex structures. This potential can be expressed in terms of the HKC metric and χ^A as

$$\chi = \frac{1}{2}\chi^A g_{AB}\chi^B\,. \tag{2.2}$$

The derivative of the hyper-Kähler potential is (locally) equal to the homothetic one-form,

$$\chi_A = \partial_A\chi\,. \tag{2.3}$$

The three covariantly constant complex structures of the hyper-Kähler cone are denoted by $\vec{J}^A{}_B$. They are hermitean, i.e. $\vec{\Omega}_{AC} \equiv g_{AB}\vec{J}^B{}_C$ is antisymmetric, and they obey the algebra of the quaternions:

$$J^\Pi J^\Sigma = -g^{\Pi\Sigma} + \varepsilon^{\Pi\Sigma}{}_\Lambda\, J^\Lambda\,, \tag{2.4}$$

which in a complex basis with components J^3 and $J^\pm = 1/2(J^1 \mp iJ^2)$, becomes

$$J^\pm\, J^3 = \pm i\, J^\pm\,, \quad (J^3)^2 = -\mathbf{1}\,, \quad (J^\pm)^2 = 0\,, \quad J^+\, J^- = -\frac{1}{2}(\mathbf{1} + iJ^3)\,. \tag{2.5}$$

Hyper-Kähler cones have an Sp(1) isometry whose Killing vectors are

$$\vec{k}^A = \vec{J}^A{}_B\, \chi^B\,. \tag{2.6}$$

To show that these are indeed Killing vectors, we note that

$$D_A \vec{k}_B = -\vec{\Omega}_{AB}\,, \tag{2.7}$$

by virtue of (2.1). The Sp(1) isometries are not triholomorphic, i.e. they do not leave the complex structures invariant. Instead the complex structures rotate under Sp(1) as

$$\mathcal{L}_{\vec{\epsilon}\cdot\vec{k}} J^{\Lambda\, A}{}_B \equiv \vec{\epsilon}\cdot \left(\vec{k}^C\, \partial_C J^{\Lambda\, A}{}_B - \partial_C \vec{k}^A\, J^{\Lambda\, C}{}_B + \partial_B \vec{k}^C\, J^{\Lambda\, A}{}_C\right) = 2\varepsilon^{\Lambda\Pi}{}_\Sigma\, \epsilon_\Pi\, J^{\Sigma\, A}{}_B\,, \tag{2.8}$$

which becomes

$$\mathcal{L}_{\vec{\epsilon}\cdot\vec{k}} J^\pm = \pm 2i(\epsilon^3\, J^\pm - \epsilon^\pm\, J^3)\,, \qquad \mathcal{L}_{\vec{\epsilon}\cdot\vec{k}} J^3 = 4i(\epsilon^+\, J^- - \epsilon^-\, J^+)\,, \tag{2.9}$$

in the complex basis. Here $\vec{\epsilon}\cdot\vec{k} = \epsilon^3 k^3 + 2(\epsilon^+ k^- + \epsilon^- k^+)$.

The hyper-Kähler potential χ is Sp(1) invariant. The four vectors associated with the homothetic conformal Killing vector, χ^A, and the three Sp(1) Killing vectors, k^{3A} and $k^{\pm A}$, define a subspace that is locally flat, i.e. the Riemann tensor vanishes when contracted with any of these four vectors. We recall from [11] that these four vectors are orthogonal (cf. (2.6)) and normalized according to

$$\chi^A\, \chi_A = k^{3A} k^3_A = 2\, k^{+A} k^-_A = 2\,\chi\,, \tag{2.10}$$

with all other inner products vanishing.

Spaces with a homothety can always be described as a cone. This becomes manifest when decomposing the coordinates ϕ^A into coordinates tangential and orthogonal to the $(4n-1)$-dimensional hypersurface defined by setting χ to a constant. The line element can then be written in the form [19],

$$\mathrm{d}s^2 = \frac{\mathrm{d}\chi^2}{2\chi} + \chi\, h_{\hat{A}\hat{B}}(x)\, \mathrm{d}x^{\hat{A}}\, \mathrm{d}x^{\hat{B}}\,, \tag{2.11}$$

where the $x^{\hat{A}}$ are the coordinates associated with the hypersurface. In the present case this hypersurface is a 3-sasakian space[3] $\mathcal{S}^{4n-1}$, and the hyper-Kähler space is therefore a cone over $\mathcal{S}^{4n-1}$. As is well known from the mathematics literature [12],

[3] For a review on 3-sasakian manifolds we refer the reader to [20]; note that $\mathcal{S}^{4n-1}$ is in general *not* the sphere S^{4n-1}.

the 3-sasakian space is an Sp(1) fibration of a $(4n-4)$-dimensional quaternion-Kähler manifold $\mathbf{Q}^{4n-4}$. Hence the manifold can be written as $R^+ \times [\mathrm{Sp}(1) \to \mathbf{S}^{4n-1} \to \mathbf{Q}^{4n-4}]$. Another relevant fibration of the quaternion-Kähler manifold is the twistor space $\mathcal{Z}$, which is a $(4n-2)$-dimensional Einstein-Kähler manifold [15, 12]. In the next subsection and in section 3, we explicitly construct this twistor space from the HKC geometry.

Some of our results of sections 3 and 4 are illustrated in a few examples, based on a $(4n)$-dimensional flat space, which is obviously a hyper-Kähler cone. In view of supergravity applications we allow for pseudo-riemannian metrics. We use complex coordinates[4] z^a, with $a = 1, \dots, 2n$, and corresponding anti-holomorphic ones $\bar{z}^{\bar{a}}$, with a metric $\eta_{a\bar{b}}$ that can be chosen diagonal with even numbers of positive and negative eigenvalues. The coordinate basis is chosen such that $J^{3\,a}{}_b = i\delta^a{}_b$. The two other complex structures, J^+ and J^-, are associated with a holomorphic and an anti-holomorphic two-form, denoted by Ω and $\bar{\Omega}$, respectively. The various quantities of interest for flat space can then be defined as

$$\begin{aligned} \chi(z,\bar{z}) &= \eta_{a\bar{b}}\, z^a \bar{z}^{\bar{b}}\,, \\ \Omega^3 &= -i\,\eta_{a\bar{b}}\,\mathrm{d}z^a \wedge \mathrm{d}\bar{z}^{\bar{b}}\,, \\ \Omega &= \frac{1}{2}\Omega_{ab}\,\mathrm{d}z^a \wedge \mathrm{d}z^b\,, \\ \bar{\Omega} &= \frac{1}{2}\bar{\Omega}_{\bar{a}\bar{b}}\,\mathrm{d}\bar{z}^{\bar{a}} \wedge \mathrm{d}\bar{z}^{\bar{b}}\,. \end{aligned} \tag{2.12}$$

The tensors Ω_{ab} are constant skew-symmetric and satisfy $\Omega_{ab}\,\bar{\Omega}_{\bar{a}\bar{b}}\,\eta^{\bar{a}b} = -\eta_{a\bar{b}}$, where $\eta^{\bar{a}b}$ is the inverse metric.

2.2 Kähler and $N = 2$ superconformal quotients

The metrics of the twistor space and the quaternion-Kähler space can be expressed directly in terms of the HKC metric by performing appropriate quotients. The resulting metric is horizontal to a certain subspace but does not come equipped with unique canonical coordinates. A choice of coordinates can be found by imposing gauge conditions associated with the isometries upon which the quotient is based. These quotients are at the heart of the superconformal multiplet calculus of supergravity [9].

In general, when the hyper-Kähler cone has an isometry with a Killing vector k^A that commutes with the dilatations, $k^A\chi_A = 0$; this implies that χ and χ_A are invariant. Hence

$$\frac{1}{\chi}\Big(g_{AB} - \frac{1}{2\chi}\chi_A\chi_B\Big) \tag{2.13}$$

is preserved by k^A. Imposing the constraint $\chi =$ constant, it follows from (2.11) that this is the 3-sasakian metric.

[4] We write capital letters $A, B, \dots = 1, 2, \dots, 4n$ for real coordinates and small letters $a, b, \dots = 1, \dots, 2n$ for holomorphic coordinates.

The quotient metric is well known; physicists find it by constructing a σ-model with (2.13) as the metric and gauging the k^A isometry by covariantizing spacetime derivatives: $\partial_\mu\phi^A \to D_\mu\phi^A = \partial_\mu\phi^A - A_\mu k^A$. The gauge field A_μ can then be eliminated by its field equation:

$$A_\mu = \frac{1}{k^B k_B}\, k_A \partial_\mu \phi^A \,. \tag{2.14}$$

Substituting this result into the σ-model action leaves the gauge invariance unaffected and has the effect of changing the metric (2.13) into

$$G_{AB} = \frac{1}{\chi}\left(g_{AB} - \frac{1}{2\chi}\,\chi_A\chi_B - \frac{1}{k^C k_C}\, k_A k_B\right). \tag{2.15}$$

Observe that this metric is horizontal in the sense that its contraction with χ^A and k^A vanishes. In the horizontal subspace, it is nondegenerate and precisely the quotient metric.

The twistor space is the quotient with respect to the k^3 isometry, and has the quotient metric:

$$G_{AB} = \frac{1}{\chi}\left(g_{AB} - \frac{1}{2\chi}\Big[\chi_A\chi_B + k^3_A k^3_B\Big]\right), \tag{2.16}$$

where we have used (2.10). Because of (2.6), k^3 is holomorphic with respect to J^3, and this is a standard Kähler quotient [21]. The moment map of the holomorphic k^3 isometry is the hyper-Kähler potential χ.

To obtain the quaternion-Kähler space, the quotient is taken with respect to the full Sp(1) isometry group. Hence one introduces gauge fields $\vec{A}_\mu$ and covariantizes the derivatives, $D_\mu\phi^A = \partial_\mu\phi^A - \vec{A}_\mu\cdot\vec{k}^A$. The field equations now yield

$$\vec{A}_\mu = \frac{1}{2}\chi^{-1}\vec{k}_A\,\partial_\mu\phi^A \,. \tag{2.17}$$

Substituting this result back into the action (which leaves the Sp(1) gauge invariance unaffected) leads to a new metric orthogonal to all four vectors χ^A and $\vec{k}^A$. This is the horizontal metric of [11]:

$$G_{AB} = \frac{1}{\chi}\left(g_{AB} - \frac{1}{2\chi}\Big[\chi_A\chi_B + \vec{k}_A\cdot\vec{k}_B\Big]\right) \,. \tag{2.18}$$

However, as the Sp(1) isometries are not triholomorphic in the hyper-Kähler cone, the above quotient not a standard hyper-Kähler quotient [17]; such quotients we call $N = 2$ *superconformal quotients*. As a result, the metric (2.18) is no longer hyper-Kähler but rather quaternion-Kähler.

It is always possible to choose a coordinate along the k^3 Killing vector; the metric of the twistor space $\mathcal{Z}$ is evidently independent of this coordinate. Consequently, the HKC metric naturally projects to the twistor space metric without the need for imposing gauge conditions. The situation regarding the quaternion-Kähler metric is different in this respect. Here we project out an $S^2 \cong \mathrm{Sp}(1)/\mathrm{U}(1)$ from the twistor space. Because there are no corresponding Killing vectors, one has to impose appropriate gauge conditions. This is discussed in section 4.

Similarly, the three quaternionic two-forms can be constructed by projecting the HKC complex structures onto the horizontal space,

$$\vec{\mathcal{Q}}_{AB} = G_{AC} \vec{J}^{C}{}_{B} \,. \tag{2.19}$$

These tensors satisfy the quaternionic algebra relations [11]

$$\begin{aligned} \mathcal{Q}^{\pm}_{AC}\, \chi\, g^{CD} \mathcal{Q}^{3}_{DB} &= \pm i \mathcal{Q}^{\pm}_{AB} \,, \qquad \mathcal{Q}^{+}_{AC}\, \chi\, g^{CD} \mathcal{Q}^{-}_{DB} = -\frac{1}{2}(G_{AB} + i\mathcal{Q}^{3}_{AB}) \,, \\ \mathcal{Q}^{3}_{AC}\, \chi\, g^{CD} \mathcal{Q}^{3}_{DB} &= -G_{AB} \,, \qquad \mathcal{Q}^{\pm}_{AC}\, \chi\, g^{CD} \mathcal{Q}^{\pm}_{DB} = 0 \,. \end{aligned} \tag{2.20}$$

Even though $\chi\, g^{AB}$ is not horizontal, it acts as an inverse metric on the horizontal subspace because it satisfies $G_{AC}\, \chi\, g^{CD} G_{DB} = G_{AB}$.

Quaternion-Kähler manifolds have non-trivial Sp(1) holonomy. In [11] the Sp(1) connection was given in terms of the Sp(1) Killing vectors of the hyper-Kähler cone,

$$\vec{\mathcal{V}}_A = \chi^{-1}\, \vec{k}_A \,. \tag{2.21}$$

This vector is invariant under the homothety and rotates under the Sp(1) isometries as a vector. Up to normalization, its pull-back is the gauge field (2.17). The curvature associated with this connection is proportional to the two-forms (2.19), as is required for a quaternion-Kähler geometry. We return to this and related points in section 4.

3. Reduction to the twistor space $\mathcal{Z}$

Consider a $4n$-dimensional hyper-Kähler cone with hyper-Kähler potential χ parametrized by $2n$ holomorphic coordinates z^a. Note that (2.1) implies that the homothetic conformal Killing vector has holomorphic and anti-holomorphic components $\chi^a(z)$ and $\chi^{\bar{a}}(\bar{z})$, respectively:

$$\chi(z,\bar{z}) = \chi^a(z)\, \chi_a(z,\bar{z}) = \chi^{\bar{a}}(\bar{z})\, \chi_{\bar{a}}(z,\bar{z}) \,, \tag{3.1}$$

where, e.g. $\chi_a(z,\bar{z}) = g_{a\bar{b}}(z,\bar{z})\, \chi^{\bar{b}}(\bar{z})$, and the metric is

$$g_{a\bar{b}}(z,\bar{z}) = \partial_a \partial_{\bar{b}} \chi(z,\bar{z}) \,. \tag{3.2}$$

The holomorphic vector field χ^a and its complex conjugate can be used to define new complex coordinates. One special coordinate is denoted by z (not to be confused with the z^a) and the $2n-1$ remaining coordinates by u^i; the precise definition of the u^i is of no concern. The u^i turn out to parametrize an Einstein-Kähler manifold, the twistor space $\mathcal{Z}$ [15]. This space is an S^2 fibration of an underlying quaternion-Kähler manifold that we discuss in section 4. The coordinate z is defined (up to a special class of holomorphic diffeomorphisms, see below) by

$$\chi^a(z,u)\,\frac{\partial}{\partial z^a} \equiv \frac{\partial}{\partial z}\,, \tag{3.3}$$

so that, upon using (2.3), (3.1) may be regarded as a first-order differential equation for χ which determines its dependence on the new coordinates z and $\bar{z}$. The result is that $\chi(z,\bar{z},u,\bar{u})$ can be written as

$$\chi(z,\bar{z};u,\bar{u}) = \mathrm{e}^{z+\bar{z}+K(u,\bar{u})}\,. \tag{3.4}$$

The function $K(u,\bar{u})$ is the Kähler potential of the Kähler quotient of the hyper-Kähler cone with respect to the U(1) isometry generated by k^3 and the compatible Kähler structure Ω^3 [21, 17]. This quotient is the twistor space $\mathcal{Z}$; it is Einstein-Kähler with metric $K_{i\bar{\jmath}}(u,\bar{u})$, and has complex dimension $2n-1$.

Observe that Kähler transformations for this twistor space, $K(u,\bar{u}) \to K(u,\bar{u}) + f(u) + \bar{f}(\bar{u})$, can be compensated by corresponding coordinate changes $z \to z - f(u)$, and hence the coordinate z is defined only modulo this ambiguity. In contrast, the hyper-Kähler potential χ in general cannot be redefined by means of a Kähler transformation because (2.1) fixes this freedom.

The HKC metric in the new coordinates (z,u^i) is

$$g_{a\bar{b}} = \partial_a\partial_{\bar{b}}\chi = \chi\begin{pmatrix} 1 & K_{\bar{\jmath}} \\ K_i & K_{i\bar{\jmath}} + K_iK_{\bar{\jmath}} \end{pmatrix}\,, \tag{3.5}$$

where we use the notation $K_i = \partial_i K(u,\bar{u})$, etc. The HKC line element takes the form

$$\mathrm{d}s^2 = \chi\Big[(\mathrm{d}z + K_i\,\mathrm{d}u^i)(\mathrm{d}\bar{z} + K_{\bar{\jmath}}\,\mathrm{d}\bar{u}^{\bar{\jmath}}) + K_{i\bar{\jmath}}\,\mathrm{d}u^i\mathrm{d}\bar{u}^{\bar{\jmath}}\Big]\,. \tag{3.6}$$

The inverse metric can be computed and equals

$$g^{\bar{a}b} = \chi^{-1}\begin{pmatrix} 1 + K^lK_l & -K^j \\ -K^{\bar{\imath}} & K^{\bar{\imath}j} \end{pmatrix}\,, \tag{3.7}$$

where $K^{\bar{\imath}j}(u,\bar{u})$ denotes the inverse of $K_{i\bar{\jmath}}$. In the following we use this metric to raise and lower indices as in $K^{\bar{\imath}} = K^{\bar{\imath}j}K_j$.

The HKC Christoffel symbols $\Gamma_{ab}{}^c = (\partial_a g_{b\bar{d}})\,g^{\bar{d}c}$ are

$$\begin{aligned}
\Gamma_{za}{}^b &= \delta_a{}^b\,,\\
\Gamma_{ij}{}^z &= K_{ij} - K_iK_j - \gamma_{ij}{}^k\,K_k\,,\\
\Gamma_{ij}{}^k &= \gamma_{ij}{}^k + 2\,K_{(i}\delta_{j)}{}^k\,,
\end{aligned} \tag{3.8}$$

where $\gamma_{ij}{}^k$ is the Christoffel connection for the twistor space $\mathcal{Z}$ and (anti) symmetrization is always done with weight one, e.g. $(ij) = 1/2(ij + ji)$.

Similarly, we compute the HKC Riemann tensor $R_{\bar{a}bc}{}^{d} = \partial_{\bar{a}}\Gamma_{bc}{}^{d}$; as the connection is independent of z and $\bar{z}$, $R_{\bar{a}zb}{}^{c} = R_{\bar{z}ab}{}^{c} = R_{\bar{a}bz}{}^{c} = 0$, i.e. the curvature vanishes when contracted with the homothetic Killing vector, as claimed in the previous section. The remaining components are

$$R_{\bar{\imath}jk}{}^{z} = -\mathcal{R}_{\bar{\imath}jk}{}^{l}\, K_l - 2\, K_{(j}K_{k)\bar{\imath}}\,, \qquad R_{\bar{\imath}jk}{}^{l} = \mathcal{R}_{\bar{\imath}jk}{}^{l} + 2\delta_{(j}{}^{l}K_{k)\bar{\imath}}\,, \tag{3.9}$$

where $\mathcal{R}_{\bar{\imath}jk}{}^{l}$ is the Riemann tensor of the twistor space $\mathcal{Z}$.

Being hyper-Kähler, the HKC is Ricci-flat, and hence the twistor space $\mathcal{Z}$ is Einstein with positive cosmological constant $2n$:

$$\mathcal{R}_{i\bar{\jmath}} = -2n\, K_{i\bar{\jmath}}\,. \tag{3.10}$$

For any Kähler manifold $R_{a\bar{b}} = \partial_a\partial_{\bar{b}} \ln\det(g_{c\bar{d}})$; for the HKC metric we have explicitly

$$\det(g_{a\bar{b}}) = \chi^{2n}\ \det(K_{i\bar{\jmath}})\,, \tag{3.11}$$

which must therefore be a product of a holomorphic and an anti-holomorphic function. Hence

$$\det(g_{a\bar{b}}) = |\mathrm{e}^{2z+f(u)}|^{2n}\,, \qquad \det(K_{i\bar{\jmath}}) = |\mathrm{e}^{f(u)}|^{2n}\, \mathrm{e}^{-2nK(u,\bar{u})}\,, \tag{3.12}$$

where $f(u)$ is the arbitrary holomorphic function that can be absorbed into the Kähler potential by performing a Kähler transformation $K(u,\bar{u}) \to K(u,\bar{u}) + f(u) + \bar{f}(\bar{u})$ on $\mathcal{Z}$. This is of course consistent with (3.10):

$$\mathcal{R}_{i\bar{\jmath}} = \partial_i\partial_{\bar{\jmath}}\ \ln[\det K_{k\bar{l}}] = -2n\, K_{i\bar{\jmath}}\,. \tag{3.13}$$

We now examine how the three Kähler forms $\vec{\Omega}$ of the hyper-Kähler cone descend to $\mathcal{Z}$. Because Ω^{+} is covariantly constant, it depends only on holomorphic coordinates, and we denote it by $\Omega_{ab}(z,u)$; similarly, $\Omega^{ab} \equiv \bar{\Omega}_{\bar{c}\bar{d}}\, g^{\bar{c}a} g^{\bar{d}b}$, which obeys $\Omega_{ac}\Omega^{cb} = J^{+}{}^{c}{}_{a} J^{-}{}^{b}{}_{c} = -\delta_a^b$, is also holomorphic. Computing the covariant derivative with respect to z, we find

$$\begin{aligned} \Omega_{ab}(z,u) &= \mathrm{e}^{2z} \begin{pmatrix} 0 & -X_j(u) \\ X_i(u) & \omega_{ij}(u) \end{pmatrix}, \\ \Omega^{ab}(z,u) &= \mathrm{e}^{-2z} \begin{pmatrix} 0 & -Y^j(u) \\ Y^i(u) & \hat{\omega}^{ij}(u) \end{pmatrix}. \end{aligned} \tag{3.14}$$

We can express $\hat{\omega}^{ij}$ and Y^i in terms of X_i, ω_{ij} and the Kähler potential and its derivatives,

$$\begin{aligned} \hat{\omega}^{ij}(u) &= \left[\omega^{ij} + 2\, K^{[i}X^{j]}\right] \mathrm{e}^{-2K}\,, \\ Y^i(u) &= \left[(1 + K_j K^j) X^i - K^i\, X^j K_j - \omega^{ij} K_j\right] \mathrm{e}^{-2K} \\ &= X^i\, \mathrm{e}^{-2K} - \hat{\omega}^{ij} K_j\,, \end{aligned} \tag{3.15}$$

where we raise and lower indices with $K^{\bar{\imath}j}$ and $K_{i\bar{\jmath}}$, $X^i = K^{i\bar{\jmath}}X_{\bar{\jmath}}$, $X_{\bar{\imath}} = (X_i)^*$, and similarly for ω^{ij}. Though it is not manifest, the right-hand sides of (3.15) are nonetheless holomorphic.

The relation $\Omega_{ac}(z,u)\,\Omega^{cb}(z,u) = -\delta_a^b$ implies the following identities,

$$\begin{aligned} X_i\, Y^i &= 1\,, \\ \omega_{ij}\, Y^j &= 0\,, \\ \hat{\omega}^{ij}\, X_j &= 0\,, \\ \hat{\omega}^{ik}\omega_{kj} &= -\delta_j^i + Y^i\, X_j\,. \end{aligned} \tag{3.16}$$

The first two equations imply that $\mathcal{L}_Y X_i = 0$, whereas the second and third display the null vectors of the odd-dimensional antisymmetric tensors ω_{ij} and $\hat{\omega}^{ij}$. Combining the above results with (3.15) leads to additional identities, such as

$$\begin{aligned} X_i X^i &= \mathrm{e}^{2K}\,, \\ \omega_{ij} X^j &= X_i\, K_j X^j - K_i\,\mathrm{e}^{2K}\,, \\ Y^i Y_i &= (1 + K_i K^i)\,\mathrm{e}^{-2K} - |Y^i K_i|^2\,, \\ X^i K_i &= Y^i K_i\,\mathrm{e}^{2K}\,. \end{aligned} \tag{3.17}$$

The structure of the HKC does not imply any constraints that do not follow from those found above.

In these coordinates, the homothetic and Sp(1) Killing vectors are

$$\begin{aligned} \chi^a &= -ik^{3a} = (1,0,\ldots,0)\,, \qquad & \chi_a &= ik_a^3 = \partial_a\chi = \chi(1,K_i)\,, \\ k_a^+ &= \Omega_{az} = \mathrm{e}^{2z}(0,X_i)\,, \qquad & k_{\bar{a}}^- &= \Omega_{\bar{a}\bar{z}}\,, \end{aligned} \tag{3.18}$$

and we may explicitly verify (2.16):

$$K_{i\bar{\jmath}} = \frac{1}{\chi}\left(g_{i\bar{\jmath}} - \frac{1}{2\chi}\Big[\chi_i\,\chi_{\bar{\jmath}} + k_i^3\,k_{\bar{\jmath}}^3\Big]\right). \tag{3.19}$$

Observe that k_a^+ is holomorphic and $k_{\bar{a}}^-$ is anti-holomorphic; raising the index of the latter one finds

$$k^{-a} = \chi^{-1}\mathrm{e}^{2\bar{z}}\,(-X^j K_j, X^i)\,, \tag{3.20}$$

where we made use of the previous identities. One can verify that the orthogonality conditions of these four vectors are indeed satisfied, e.g. $k^{-a}k_a^+ = \chi$. Furthermore one can verify that $D_a k_b^+ = -\Omega_{ab}$ and $D_{\bar{a}}k_{\bar{b}}^- = -\bar{\Omega}_{\bar{a}\bar{b}}$ as specified by (2.7). This leads directly to

$$D_i X_j = -\omega_{ij} + 2\,K_{(i}X_{j)}\,, \tag{3.21}$$

where D_i contains only the Christoffel connection $\gamma_{ij}{}^k$ of $\mathcal{Z}$; the second term is due to the extra term in the hyper-Kähler connection $\Gamma_{ij}{}^k$. Hence it follows that ω_{ij} is (locally) exact. The pair X, ω is a contact structure. The result (3.21) is also required by covariant constancy of Ω_{ab}. However, though (3.21) implies $D_{(i}(\mathrm{e}^{-2K}X_{j)}) = 0$, $\mathrm{e}^{-2K}X_i$ is not a Killing vector of the twistor space, as $\partial_{(\bar{\imath}}(\mathrm{e}^{-2K}X_{j)}) \neq 0$. We return to this point in the next section where we discuss the role played by X^i for the quaternion-Kähler space.

All hyper-Kähler cones have a homothety and Sp(1) isometries; in some cases, they may have additional isometries. A triholomorphic isometry leaves the complex structures invariant. Not all HKC isometries descend to isometries of the twistor space; for example, the $k^{\pm}$ isometries are *not* isometries of the twistor space. HKC isometries that commute with the homothety and the k^3 isometry do not depend on the coordinate z, and *do* descend to isometries on the twistor space. A general analysis of the HKC Killing equation leads to the following form for HKC Killing vectors:

$$k^z = i(K_i\mu^i - \mu)\,, \qquad k^i = -i\mu^i\,, \tag{3.22}$$

where $\mu(u,\bar{u})$ is a real function on $\mathcal{Z}$, with $\mu_i = \partial_i\mu$ and $\mu^i = \mu_{\bar{\jmath}}K^{\bar{\jmath}i}$, satisfying

$$D_i\partial_j\mu = 0\,. \tag{3.23}$$

Other HKC isometries depend explicitly on z (it turns out that they can be encoded in a holomorphic function and a holomorphic one-form on the twistor space) and, with the exception of the Sp(1) isometries, are disregarded in what follows (physically, they cannot be gauged by coupling to an $N = 2$ vector multiplet). From (3.23) one can prove that the vector (3.22) is holomorphic. Furthermore it follows that the hyper-Kähler potential χ is invariant whereas $K(u,\bar{u})$ changes by a Kähler transformation. Hence the twistor space $\mathcal{Z}$ admits an isometry generated by

$$k_i = i\,\partial_i\mu \tag{3.24}$$

and its complex conjugate. The Killing equation on $\mathcal{Z}$ can be verified directly from (3.23). We note that the moment map of this isometry is the function μ itself. The case of constant μ corresponds to the k^3 isometry, which acts trivially on $\mathcal{Z}$.

If, in addition, the isometry (3.22) is triholomorphic in the hyper-Kähler cone, then there is an extra constraint on μ:

$$\mathcal{L}_k X_i \equiv -i\mu^j\partial_j X_i - i\partial_i\mu^j\,X_j = -2i(K_j\mu^j - \mu)X_i\,; \tag{3.25}$$

equivalently, $\hat{\omega}_{ij}\mu^j\mathrm{e}^{2K} + X_j\,D_i\,\mu^j + 2X_i\,\mu = 0$. Triholomorphic HKC isometries thus always descend to holomorphic isometries on $\mathcal{Z}$.

To end this section, we turn to the flat hyper-Kähler cone whose quantities of interest were defined in (2.12), and demonstrate explicitly that the corresponding twistor spaces are the complex projective spaces CP^{2n-1} (or their noncompact versions). We start by singling out two of the complex coordinates with positive metric, say z^{2n} and z^{2n-1}, and bring the hyper-Kähler potential and the holomorphic two-form into the form

$$\begin{aligned}\chi(z,\bar z) &= \sum_{i,j=1}^{2n-2} \eta_{i\bar\jmath}\, z^i \bar z^{\bar\jmath} + z^{2n-1}\bar z^{2n-1} + z^{2n}\bar z^{2n}\,,\\ \Omega &= \frac{1}{2}\omega_{ij}\,\mathrm{d}z^i\wedge\mathrm{d}z^j + \Omega_{iz}\,\mathrm{d}z^i\wedge\mathrm{d}z^{2n} \equiv \sum_{i=1}^{n}\mathrm{d}z^{2i-1}\wedge\mathrm{d}z^{2i}\,. \end{aligned}\tag{3.26}$$

We now substitute

$$z^{2n} = \mathrm{e}^z\,,\qquad z^i = \mathrm{e}^z\,u^i\,,\qquad (i=1,\ldots,2n-1)\tag{3.27}$$

and find

$$\begin{aligned}\chi(z,\bar z,u,\bar u) &= \exp[z+\bar z+K(u,\bar u)]\,,\\ \Omega &= \mathrm{e}^{2z}\Big[\frac{1}{2}\,\omega_{ij}\,\mathrm{d}u^i\wedge\mathrm{d}u^j - (\omega_{ij}u^j+\Omega_{iz})\,\mathrm{d}z\wedge\mathrm{d}u^i\Big]\,,\end{aligned}\tag{3.28}$$

where

$$K(u,\bar u) = \ln\Big[1+\sum_{i,j=1}^{2n-1}\eta_{i\bar\jmath}\,u^i\,\bar u^{\bar\jmath}\Big]\,.\tag{3.29}$$

For $\eta_{i\bar\jmath}=\delta_{i\bar\jmath}$, the Kähler potential $K(u,\bar u)$ of the twistor space $\mathcal{Z}$ is the Kähler potential of CP^{2n-1} (similarly for the indefinite case) and the metric is

$$K_{i\bar\jmath} = \frac{1}{1+\eta_{m\bar n}\,u^m\,\bar u^{\bar n}}\left(\eta_{i\bar\jmath} - \frac{\eta_{i\bar k}\,\bar u^{\bar k}\,u^l\,\eta_{l\bar\jmath}}{1+\eta_{p\bar q}\,u^p\,\bar u^{\bar q}}\right)\,.\tag{3.30}$$

The determinant of the metric is

$$\det(K_{i\bar\jmath}) = \det(\eta_{i\bar\jmath})\,[1+\eta_{m\bar n}\,u^m\,\bar u^{\bar n}]^{-2n} = \det(\eta_{i\bar\jmath})\,\mathrm{e}^{-2n\,K(u.\bar u)}\,,\tag{3.31}$$

in accord with (3.12). The inverse metric is

$$K^{\bar\imath j} = (1+\eta_{m\bar n}\,u^m\,\bar u^{\bar n})\,[\eta^{\bar\imath j}+\bar u^{\bar\imath}u^j]\,,\tag{3.32}$$

where $\eta^{\bar\imath j}$ is the inverse of $\eta_{i\bar\jmath}$. From the holomorphic two-form in these coordinates one can read off the holomorphic one-form

$$X = (\omega_{ij}u^j+\Omega_{iz})\,\mathrm{d}u^i \equiv \sum_{i=1}^{n-1}(u^{2i}\mathrm{d}u^{2i-1}-u^{2i-1}\mathrm{d}u^{2i}) + \mathrm{d}u^{2n-1}\,.\tag{3.33}$$

Finally one can easily verify that $Y^i = \delta^{i,2n-1}$, and that all relations in (3.16) and (3.17) are satisfied.

4. Quaternion-Kähler geometry

In this section, we construct the quaternion-Kähler manifold $\mathbf{Q}^{4(n-1)}$ and its geometry. We begin by writing the horizontal metric and two-forms of the hyper-Kähler cone [11] in the special coordinates u^i, z introduced in the previous section. The metric $G_{a\bar b}$ (2.18) becomes

$$G_{i\bar\jmath} = K_{i\bar\jmath} - \mathrm{e}^{-2K} X_i X_{\bar\jmath}\,, \qquad G_{z\bar\imath} = G_{z\bar z} = 0\,, \tag{4.1}$$

which is manifestly orthogonal to the homothetic Killing vector and the Sp(1) Killing vectors k^3 and $k^{\pm i}$, and hence

$$X^i G_{i\bar\jmath} = X^{\bar\imath} G_{i\bar\jmath} = 0\,. \tag{4.2}$$

Moreover, the horizontal metric G is invariant under the Sp(1) diffeomorphisms:

$$(\mathcal{L}_{k^\pm} G)_{i\bar\jmath} = (\mathcal{L}_{k^\pm} G)_{ij} = 0\,. \tag{4.3}$$

Because $G_{i\bar\jmath}$ is z-independent and $G_{z\bar\imath} = 0$, the Lie derivative involves only the components $k^{\pm i}$. This identity follows from the equations (3.15), (3.16), (3.21) and (4.2).

The horizontal two-forms $\vec{\mathcal{Q}}_{AB}$ (2.19) become

$$\mathcal{Q}^3_{i\bar\jmath} = -i\, G_{i\bar\jmath}\,, \qquad \mathcal{Q}^3_{i\bar z} = \mathcal{Q}^3_{z\bar\imath} = \mathcal{Q}^+_{zi} = 0\,, \qquad \mathcal{Q}^+_{ij} = \mathrm{e}^{z-\bar z}\mathrm{e}^K\,\hat\omega_{ij}\,, \tag{4.4}$$

and satisfy

$$X^i \mathcal{Q}^3_{i\bar\jmath} = \mathcal{Q}^3_{i\bar\jmath} X^{\bar\jmath} = X^i \mathcal{Q}^+_{ij} = 0\,. \tag{4.5}$$

These two-forms rotate under Sp(1) according to

$$\mathcal{L}_{k^-}\mathcal{Q}^+ = -i\mathcal{Q}^3\,, \qquad \mathcal{L}_{k^+}\mathcal{Q}^3 = -2i\mathcal{Q}^+\,. \tag{4.6}$$

Note that the z-dependence of Q^+_{ij} is relevant here, as z transforms under the k^- isometry.

The $\vec{\mathcal{Q}}$ are not covariantly constant in the twistor space but satisfy the relations

$$\begin{aligned} D_k \mathcal{Q}^3_{i\bar\jmath} &= i\mathcal{V}^-_{\bar\jmath}\,\mathcal{Q}^+_{ik}\,,\\ D_k \mathcal{Q}^+_{ij} &= i\mathcal{V}^3_k\,\mathcal{Q}^+_{ij}\,,\\ D_k \mathcal{Q}^-_{\bar\imath\bar\jmath} &= -i\mathcal{V}^3_k\,\mathcal{Q}^-_{\bar\imath\bar\jmath} + 2i\mathcal{V}^-_{[\bar\imath}\,\mathcal{Q}^3_{\bar\jmath]k}\,, \end{aligned} \tag{4.7}$$

where the Sp(1) connections $\vec{\mathcal{V}}$ were defined in (2.21) and the covariant derivatives are defined with the twistor space affine connection $\gamma_{ij}{}^k$. In [11] it was shown that there exists another affine connection with respect to which the $\vec{\mathcal{Q}}$ are Sp(1) covariantly constant. This connection projects to the affine connection of $\mathbf{Q}^{4(n-1)}$.

JHEP02(2001)039

Now we project the metric and two-forms onto the quaternion-Kähler manifold $\mathbf{Q}^{4(n-1)}$. This space can be described as the subspace of the twistor space $\mathcal{Z}$ orthogonal to the vector X^i. As X^i is neither holomorphic nor Killing, we cannot perform a quotient[5]. Fortunately, there is a holomorphic vector field $Y^i(u)$, that we can use to single out a suitable coordinate ζ, and we define

$$Y^i(v,\zeta)\,\frac{\partial}{\partial u^i} \equiv \frac{\partial}{\partial \zeta}\,. \tag{4.8}$$

This choice of ζ is canonical but not unique.

In this way we decompose the holomorphic coordinates u^i of the Einstein-Kähler manifold into a special holomorphic coordinate ζ and $2n-2$ remaining ones v^α, such that the $(4n-4)$-dimensional manifold parametrized by the coordinates v^α and $\bar{v}^{\bar{\alpha}}$ is quaternion-Kähler. However, we still have to fix the dependence on the coordinates $z-\bar{z}$ and ζ by choosing a suitable Sp(1) gauge condition.

In the new coordinates (v^α,ζ) the vector $Y^i=(0,\ldots,0,1)$. It then follows from the first equation of (3.16) that $X_\zeta=1$. Using (3.21) and (3.16) we then derive that

$$\partial_\zeta X_i = Y^j\,\partial_j X_i = Y^j\,\partial_i X_j = 0\,. \tag{4.9}$$

These results are summarized by $(\alpha=1,\ldots,2n-2)$,

$$Y^\zeta = 1\,,\quad Y^\alpha = 0\,,\quad X_\zeta = 1\,,\quad X_\alpha = X_\alpha(v)\,. \tag{4.10}$$

The identities (3.17) now lead to five more (dependent) relations. Defining

$$Z^\alpha = \hat{\omega}^{\alpha\zeta}\,, \tag{4.11}$$

these relations read as follows,

$$\begin{aligned} \omega_{\alpha\zeta} &= 0\,,\\ \hat{\omega}^{\alpha\gamma}\,\omega_{\gamma\beta} &= -\delta^\alpha_\beta\,,\\ Z^\alpha\,X_\alpha &= 0\,,\\ \omega_{\alpha\beta}\,Z^\beta &= X_\alpha\,,\\ \hat{\omega}^{\alpha\beta}\,X_\beta &= -Z^\alpha\,. \end{aligned} \tag{4.12}$$

Subsequently one proves that

$$\partial_\zeta\omega_{\alpha\beta} = -2\partial_{[\alpha}\omega_{\beta]\zeta} = 0\,, \tag{4.13}$$

so that $\omega_{\alpha\beta}$ does not depend on ζ.

[5] If the latter were a holomorphic vector field, a canonical way to project to the horizontal subspace would be to define a coordinate ζ by $X^i\partial_i \equiv \partial_\zeta$, in the same way as for the k^3 isometry (c.f. (3.3)). After performing the quotient, the metric $G_{i\bar{\jmath}}$ would then become horizontal with respect to the ζ-direction and independent of ζ because of (4.3), and the quaternion-Kähler space would be obtained directly.

The above equations then show that Z^α and $\hat\omega^{\alpha\beta}$ are also independent of ζ, so that we obtain the following decompositions for the HKC holomorphic tensors Ω_{ab} and Ω^{ab},

$$\begin{aligned}
\Omega_{ab}(z,v) &= \mathrm{e}^{2z}\begin{pmatrix} \omega_{\alpha\beta}(v) & 0 & X_\alpha(v) \\ 0 & 0 & 1 \\ -X_\beta(v) & -1 & 0 \end{pmatrix}\,, \\
\Omega^{ab}(z,v) &= \mathrm{e}^{-2z}\begin{pmatrix} \hat\omega^{\alpha\beta}(v) & Z^\alpha(v) & 0 \\ -Z^\beta(v) & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\,,
\end{aligned} \tag{4.14}$$

where

$$\omega_{\alpha\beta}(v) = -\partial_{[\alpha}X_{\beta]}(v)\,. \tag{4.15}$$

Observe that these tensors are thus entirely expressed in terms of the $X_\alpha(v)$.

For later use, we note the following identities, which follow from the second equation in (3.15),

$$\begin{aligned}
X^\alpha &= (\hat\omega^{\alpha\beta}K_\beta + Z^\alpha K_\zeta)\,\mathrm{e}^{2K}\,, \\
X^\zeta &= (1 - Z^\alpha K_\alpha)\,\mathrm{e}^{2K}\,.
\end{aligned} \tag{4.16}$$

We now construct the quaternion-Kähler metric and two-forms from the horizontal metric and two-forms by imposing Sp(1) gauge conditions.[6] A convenient set of conditions is $z - \bar z = \zeta = 0$. This gauge can indeed be chosen by using the remaining $k^{-i} \propto X^i$ symmetry, since X^ζ is generically non-vanishing. As discussed above, because ζ is not a coordinate along a Killing vector, the metric $G_{i\bar\jmath}$ is in general not independent of ζ and $G_{\zeta\bar\imath}$ is non-zero. Hence, the quaternion-Kähler metric is obtained by setting $\zeta = 0 \Rightarrow \mathrm{d}\zeta = 0$. Since this is a holomorphic gauge choice, the metric is still hermitean, and its components are given by

$$G_{\alpha\bar\beta} = K_{\alpha\bar\beta} - \mathrm{e}^{-2K}X_\alpha X_{\bar\beta}\,. \tag{4.17}$$

This metric is non-degenerate; its inverse can be expressed in terms of $K^{\bar\alpha\beta} = (K_{\alpha\bar\beta})^{-1}$,

$$G^{\bar\alpha\beta} = K^{\bar\alpha\beta} + \mathrm{e}^{-2K}\frac{K^{\bar\alpha\gamma}X_\gamma K^{\bar\delta\beta}X_{\bar\delta}}{1 - \mathrm{e}^{-2K}X_{\bar\delta}K^{\bar\delta\gamma}X_\gamma}\,. \tag{4.18}$$

[6]As the Sp(1) isometry of the hyper-Kähler cone is lost when descending to the twistor space $\mathcal{Z}$, one may wonder how a *gauge* symmetry corresponding to $k^\pm$ can act on the twistor space. Mathematically, this happens because the $S^2 \equiv \mathrm{Sp}(1)/\mathrm{U}(1)$ bundle is non-trivial. Physically, if we gauge the full Sp(1) in the hyper-Kähler cone, after eliminating the U(1) connection (to descend to $\mathcal{Z}$), the remaining connections corresponding to the coset generators $k^\pm$ have gauge transformations that include a term proportional to the U(1) connection. Since this connection is determined in terms of the coordinates of $\mathcal{Z}$, we can compute its curvature; we find the Kähler form of $\mathcal{Z}$. Consequently, this inhomogeneous term in the transformations of the coset connections is an obstruction to finding a rigid isometry, but clearly we can still choose a *local* gauge fixing condition for the coset generators.

The quaternion-Kähler two-forms are given by

$$\mathcal{Q}^3_{\alpha\bar\beta} = -iG_{\alpha\bar\beta} \qquad \mathcal{Q}^+_{\alpha\beta} = \mathrm{e}^{-K}(\omega_{\alpha\beta} + 2K_{[\alpha}X_{\beta]})\,. \tag{4.19}$$

We suppress the + superscript on the tensor $\mathcal{Q}^+_{\alpha\beta}$ below, as its holomorphic indices indicate that we are dealing with $\mathcal{Q}^+$. The two-form $\mathcal{Q}$ should be non-degenerate, such that there exist an inverse $\mathcal{Q}^{\alpha\beta}$,

$$\mathcal{Q}^{\alpha\gamma}\mathcal{Q}_{\gamma\beta} = -\delta^\alpha{}_\beta\,. \tag{4.20}$$

This inverse tensor can be found explicitly. Using (4.12) and (4.16), one can verify that it takes the form

$$\mathcal{Q}^{\alpha\beta} = \mathrm{e}^K\left[\hat\omega^{\alpha\beta} - 2\frac{Z^{[\alpha}X^{\beta]}}{X^\zeta}\right], \tag{4.21}$$

and that it is related to $\mathcal{Q}^- \equiv \bar{\mathcal{Q}}$ by

$$\mathcal{Q}^{\alpha\beta} = \bar{\mathcal{Q}}_{\bar\gamma\bar\delta}\,G^{\bar\gamma\alpha}\,G^{\bar\delta\beta}\,. \tag{4.22}$$

This property ensures that the quaternionic algebra holds. Note that the expression for the inverse is non-degenerate when X^ζ is nonvanishing. According to (4.16), this is so when $Z^\alpha K_\alpha \neq 1$, which is generically the case because Z^α is holomorphic and K_α is not.

The Sp(1) connections of the quaternion-Kähler space follow from (2.21),

$$\mathcal{V}^3_\alpha = -iK_\alpha\,, \qquad \mathcal{V}^+_\alpha = \mathrm{e}^{-K}\,X_\alpha\,, \tag{4.23}$$

and their complex conjugates. The Sp(1) curvature two-forms are then defined in the quaternion-Kähler space by

$$\mathcal{R}^3 \equiv \mathrm{d}\mathcal{V}^3 - 2i\mathcal{V}^+\wedge\mathcal{V}^-\,, \qquad \mathcal{R}^+ \equiv \mathrm{d}\mathcal{V}^+ - i\mathcal{V}^3\wedge\mathcal{V}^+\,. \tag{4.24}$$

and satisfy

$$\vec{\mathcal{Q}} = -\frac{1}{2}\vec{\mathcal{R}}\,. \tag{4.25}$$

These formulae can be derived both in the twistor and quaternion-Kähler spaces because the Bianchi identities,

$$\mathrm{d}\mathcal{R}^3 = 2i(\mathcal{V}^+\wedge\mathcal{R}^- - \mathcal{V}^-\wedge\mathcal{R}^+)\,, \qquad \mathrm{d}\mathcal{R}^+ = i(\mathcal{V}^3\wedge\mathcal{R}^+ - \mathcal{V}^+\wedge\mathcal{R}^3)\,, \tag{4.26}$$

hold in both cases (see (4.7)).

In section 4 we have shown that triholomorphic HKC isometries descend to holomorphic isometries on the twistor space $\mathcal{Z}$, with the additional constraint (3.25). We now study how these isometries descend to the quaternion-Kähler manifold and how they give rise to quaternionic isometries (i.e. isometries that leave the quaternionic structure invariant up to an Sp(1) rotation). The fact that the triholomorphic HKC isometries lead to quaternionic isometries is known from the mathematics literature [12], and can be understood from the superconformal calculus [11].

JHEP02(2001)039

We start by observing that $G_{i\bar{j}}$ is preserved by the triholomorphic isometry (3.24). Indeed, using (3.25), we find that

$$\mathcal{L}_k G_{i\bar{j}} = 0\,. \tag{4.27}$$

Here we take the Lie derivative along the total vector field k comprising both k^i and $k^{\bar{\imath}}$. Obviously, the action of this isometry is not in general restricted to the quaternion-Kähler subspace, because the coordinate ζ, which has been put to zero by the gauge choice, may change. To correct for this we have to add a *compensating* Sp(1) transformation (with a coordinate-dependent coefficient) to restore the $\zeta = 0$ gauge. Because the Sp(1) transformation takes the form $\delta u^i \propto X^i$, we must thus combine the action of the isometry associated to $k_i = i\partial_i\mu \equiv i\mu_i$ (see (3.24)) with the following compensating Sp(1) transformation:

$$\delta\zeta = -k^\zeta = i\mu^\zeta\,, \qquad \delta u^\alpha = -\frac{k^\zeta}{X^\zeta}X^\alpha = i\frac{\mu^\zeta}{X^\zeta}X^\alpha\,. \tag{4.28}$$

This modification leaves $G_{i\bar{j}}$ invariant because $X^i G_{i\bar{j}} = 0$ and hence the Lie derivative of the metric $G_{i\bar{j}}$ along fX^i for *any* function f of the coordinates (see (4.3)) vanishes. Thus we conclude that the vector defined by

$$\hat{k}^\alpha = k^\alpha - \frac{k^\zeta}{X^\zeta}X^\alpha = -i\Big[\mu^\alpha - \mu^\zeta\,\frac{X^\alpha}{X^\zeta}\Big]\,, \qquad \hat{k}^\zeta = 0\,, \tag{4.29}$$

preserves the gauge $\zeta = 0$, and therefore defines an isometry of the quaternion-Kähler metric:

$$\mathcal{L}_{\hat{k}} G_{\alpha\bar{\beta}} = 0\,. \tag{4.30}$$

A similar result can be derived for the action of the isometries on the quaternionic structure. Following the same procedure as for the metric, we first determine the variation with respect to the isometry (3.24) of $\mathcal{Q}^3_{i\bar{j}}$ and $\mathcal{Q}^+_{ij}$ (see (4.4)), and find (provided we also take the variations of z and $\bar{z}$ into account) that they are invariant:

$$\mathcal{L}_k \vec{\mathcal{Q}} = 0\,. \tag{4.31}$$

However, these isometries do not preserve the gauge $z - \bar{z} = \zeta = 0$. Hence we must introduce an infinitesimal compensating Sp(1) transformation to restore the gauge conditions, one as in (4.28) for $\zeta = 0$ and a similar one for $z - \bar{z} = 0$. The combined effect of the projected HKC isometry and the compensating Sp(1) transformation rotates the $\vec{\mathcal{Q}}$ by an Sp(1) rotation. Restricting the twistor space forms $\vec{\mathcal{Q}}$ to the quaternion-Kähler ones (see (4.19)), we thus derive the following result for the quaternionic structure,

$$\begin{aligned}
\mathcal{L}_{\hat{k}}\mathcal{Q}^3 &= -2\,\mathrm{e}^K\Big[\frac{\mu^{\bar{\zeta}}}{\bar{X}^{\bar{\zeta}}}\,\mathcal{Q}^+ + \frac{\mu^\zeta}{X^\zeta}\,\mathcal{Q}^-\Big]\,, \\
\mathcal{L}_{\hat{k}}\mathcal{Q}^+ &= -i\Big[K_i(\mu^i - \mu^\zeta\frac{X^i}{X^\zeta}) + K_{\bar{\imath}}(\mu^{\bar{\imath}} - \mu^{\bar{\zeta}}\frac{X^{\bar{\imath}}}{X^{\bar{\zeta}}}) - 2\mu\Big]\,\mathcal{Q}^+ + \frac{\mathrm{e}^K\,\mu^\zeta}{X^\zeta}\,\mathcal{Q}^3\,.
\end{aligned} \tag{4.32}$$

To end this section it is instructive to return to the example based on a hyper-Kähler cone with a flat $(4n)$-dimensional (pseudo)-riemannian metric. The twistor space $\mathcal{Z}$ associated with the underlying quaternion-Kähler manifold is a (noncompact) CP^{2n-1} and was discussed in the previous section. Now we construct the underlying quaternion-Kähler manifold. At the end of the previous section, we found that the coordinate $\zeta = u^{2n-1}$. It is now straightforward to determine the metric from (4.17) by restricting the coordinates to the remaining ones, denoted by u^α with $\alpha = 1, \ldots, 2n-2$, and putting $\zeta = 0$. The result is

$$G_{\alpha\bar\beta} = \frac{\eta_{\alpha\bar\beta}}{1+\eta_{\gamma\bar\delta}\, v^\gamma\, \bar v^{\bar\delta}} - \frac{\eta_{\alpha\bar\gamma}\bar v^{\bar\gamma}\, v^\delta \eta_{\delta\bar\beta} - \omega_{\alpha\gamma} v^\gamma\, \bar v^{\bar\delta} \bar\omega_{\bar\delta\bar\beta}}{(1+\eta_{\gamma\bar\delta}\, v^\gamma\, \bar v^{\bar\delta})^2}\,. \tag{4.33}$$

This is the metric for (noncompact) $HP(n-1)$. The Sp(1) curvature can also easily be computed,

$$\mathcal{R}^+ = -\left[\frac{\omega_{\alpha\beta}}{1+\eta_{\gamma\bar\delta}\, v^\gamma\, \bar v^{\bar\delta}} - 2\,\frac{\eta_{\alpha\bar\gamma}\bar v^{\bar\gamma}\, v^\delta \omega_{\delta\beta}}{(1+\eta_{\gamma\bar\delta}\, v^\gamma\, \bar v^{\bar\delta})^2}\right] \mathrm{d}v^\alpha \wedge \mathrm{d}v^\beta\,. \tag{4.34}$$

A particular case of this series is the four-sphere $S^4 = HP^1$, so $n=2$. As this space is compact, we take $\eta_{\alpha\bar\beta} = \delta_{\alpha\bar\beta}$ and $\omega_{12} = 1$. The metric simplifies to

$$\mathrm{d}s^2 = \frac{\mathrm{d}v^\alpha\, \mathrm{d}\bar v^{\bar\alpha}}{(1+|v|^2)^2}\,, \tag{4.35}$$

Which is indeed the conformally flat metric on S^4. The Sp(1) curvature takes the simple form

$$\mathcal{R}^3 = 2i\frac{\mathrm{d}v^\alpha \wedge \mathrm{d}\bar v^{\bar\alpha}}{(1+|v|^2)^2}\,, \qquad \mathcal{R}^+ = -2\,\frac{\mathrm{d}v^1 \wedge \mathrm{d}v^2}{(1+|v|^2)^2}\,. \tag{4.36}$$

For the noncompact case, namely four-dimensional anti-de-Sitter space, we choose $\eta_{\alpha\bar\beta} = -\delta_{\alpha\bar\beta}$. The metric components then are, with $v^1 = u$ and $v^2 = v$,

$$G_{u\bar u} = G_{v\bar v} = -\frac{1}{1-u\bar u - v\bar v}\,, \qquad G_{u\bar v} = 0\,. \tag{4.37}$$

5. Superconformal tensor multiplets

Metrics on $4n$ (real) dimensional hyper-Kähler cones with n commuting triholomorphic isometries can be constructed (locally) by a duality transformation of a general superconformally invariant $N=2$ tensor multiplet action. Every hyper-Kähler cone with such isometries can be obtained in this way.

An $N=2$ tensor supermultiplet consists of three scalar fields, which we group into a real scalar x and a complex one v, and a tensor gauge field $B_{\mu\nu}$ with corresponding gauge-invariant field strength $H^\mu = -1/2\varepsilon^{\mu\nu\rho\sigma}\,\partial_\nu B_{\rho\sigma}$. Furthermore it contains a doublet of Majorana spinors and its (minimal) off-shell version requires an auxiliary complex scalar field. The first part of the discussion below closely follows [22, 17].

To obtain a hyper-Kähler cone one starts from a system of hypermultiplets that is invariant under (rigid) superconformal transformations. However, for the moment we ignore the superconformal aspects and proceed to describe the most general couplings of tensor multiplets. The restrictions imposed by superconformal invariance are introduced below. In $N=1$ superspace, a general lagrangian for tensor multiplets is a real function $F(x^I, v^I, \bar v^I)$, where indices $I, J, \ldots$ label the n tensor multiplets, such that F satisfies the following differential equation:

$$F_{x^I x^J} + F_{v^I \bar v^J} = 0\,. \tag{5.1}$$

Here the subscripts denote differentiation with respect to the corresponding fields. Observe that this constraint implies that the mixed derivative with respect to v^I and $\bar v^J$ is symmetric in the indices I and J.

The bosonic part of the lagrangian for the n tensor supermultiplets is

$$\mathcal{L} = F_{x^I x^J}\Big(\partial_\mu v^I \partial^\mu \bar v^I + \frac{1}{4}(\partial_\mu x^I \partial^\mu x^J - H^I_\mu H^{\mu J})\Big) + \frac{1}{2} i\Big(F_{v^I x^J}\partial_\mu v^I - F_{\bar v^I x^J}\partial_\mu \bar v^I\Big) H^{\mu J}\,. \tag{5.2}$$

For fixed x^I, the corresponding nonlinear sigma model is a Kähler space with Kähler potential equal to $F(v^I, \bar v^I)$, whereas for fixed v, one gets the bosonic part of the supersymmetric lagrangian for $N=1$ tensor multiplets. After adding a total derivative, the term linear in $H^{\mu I}$ can be rewritten as

$$\mathcal{L}' = \frac{1}{4} i\Big(F_{v^I x^J x^K}\partial_\mu v^I \partial_\nu x^K - F_{\bar v^I x^J x^K}\partial_\mu \bar v^I \partial_\nu x^K - 2F_{x^I x^J x^K}\partial_\mu \bar v^I \partial_\nu v^K\Big)\varepsilon^{\mu\nu\rho\sigma}\, B^J_{\rho\sigma}\,. \tag{5.3}$$

We now dualize the vectors $H^{\mu I}$ by introducing real multipliers y_I, and adding to the action a term $-1/2 y_I \partial_\mu H^{\mu I}$. The field equations for the y_I now ensure that the quantities $H^{\mu I}$ satisfy the constraint $\partial_\mu H^{\mu I} = 0$. However, rather then imposing the field equations for the y_I one can solve the field equations for the $H^{\mu I}$. This yields the lagrangian,

$$\begin{aligned} \mathcal{L} = {}& F_{x^I x^J}\Big(\partial_\mu v^I \partial^\mu \bar v^J + \frac{1}{4}\partial_\mu x^I \partial^\mu x^J\Big) + \\ & + \frac{1}{4} F^{x^I x^J}\Big(\partial_\mu y_I + i(\partial_\mu v^K F_{v^K x^I} - F_{x^I \bar v^K}\partial_\mu \bar v^K)\Big) \times \\ & \times \Big(\partial^\mu y_J + i(\partial^\mu v^L F_{v^L x^J} - F_{x^J \bar v^L}\partial^\mu \bar v^L)\Big)\,, \end{aligned} \tag{5.4}$$

where $F^{x^I x^J}$ denotes the inverse of $F_{x^I x^J}$.

At this point we introduce a second set of n complex fields w_I by the conditions

$$w_I = \frac{1}{2}\Big(i y_I + \frac{\partial F}{\partial x^I}\Big)\,, \tag{5.5}$$

which determine x^I and y_I in terms of the v^I, w_I, $\bar v^I$ and $\bar w_I$. The definition (5.5) may seem somewhat arbitrary but is obvious when performing the duality transformation in terms of $N=1$ superfields. Note that the metric describing the target space of the nonlinear sigma model with coordinates v^I and w_I is independent of $w_I - \bar{w}_I$, and hence the space has n commuting isometries associated with shifts of w_I by imaginary constants. Moreover, as the action is $N=2$ supersymmetric, the space is hyper-Kähler.

To evaluate the metric in terms of the complex coordinates, we vary (5.5) and find

$$\delta x^I = F^{x^I x^J}\left((\delta w_J + \delta\bar{w}_J) - \delta v^K F_{v^K x^J} - F_{x^J \bar{v}^K}\delta\bar{v}^K\right). \tag{5.6}$$

With the help of this equation the metric follows straightforwardly:

$$\begin{aligned}
g_{v^I\bar{v}^J} &= -\Big(F_{x^I x^J} + F_{v^I x^K} F^{x^K x^L} F_{x^L \bar{v}^J}\Big)\,,\\
g_{v^I \bar{w}_J} &= F_{v^I x^K} F^{x^K x^J}\,,\\
g_{w_I \bar{v}^J} &= F^{x^I x^K} F_{x^K \bar{v}^J}\,,\\
g_{w_I \bar{w}_J} &= -F^{x^I x^J}\,.
\end{aligned} \tag{5.7}$$

The inverse metric can be computed directly and its components are

$$\begin{aligned}
g^{\bar{v}^I v^J} &= -F^{x^I x^J}\,,\\
g^{\bar{v}^I w_J} &= -F^{x^I x^K} F_{x^K v^J}\,,\\
g^{\bar{w}_I v^J} &= -F_{\bar{v}^I x^K} F^{x^K x^J}\,,\\
g^{\bar{w}_I w_J} &= -(F_{x^I x^J} + F_{\bar{v}^I x^K} F^{x^K x^L} F_{x^L v^J})\,.
\end{aligned} \tag{5.8}$$

Finally one easily verifies that this space is a Kähler space with Kähler potential equal to

$$K(v,w,\bar{v},\bar{w}) = F(x,v,\bar{v}) - (w_I + \bar{w}_I)\, x^I\,. \tag{5.9}$$

For further discussion of the geometric properties of this hyper-Kähler space we refer to [17].

We now discuss the superconformal couplings of the n tensor multiplets. The scaling weight of the tensor gauge field is fixed so that it is compatible with its gauge invariance, and hence the field strengths scale with weight 3. To have scale invariance for the full action the scalar fields x^I and v^I must therefore have weight 2, whereas the second derivatives of the function F should scale with weight -2 such that the lagrangian scales uniformly with weight 4 (recall a spacetime derivative has weight $+1$). Therefore it follows that the derivatives $F_{x^I x^J}$ and $F_{x^I v^J}$ must be homogeneous functions of x^I, v^I and $\bar{v}^I$ of degree -1. This condition is not yet sufficient for superconformal invariance, because we know from $N=1$ superconformal symmetry that the theory must also be invariant under phase transformations of the complex fields v^I. Hence $N=2$ superconformal invariance requires both (5.1) and the identities (A.1) presented in appendix A; these are implied by the invariance and homogeneity requirements discussed above.

Subsequently, in appendix A, we analyze the identities (A.1) and show that, modulo irrelevant terms in F that do *not* contribute to the action, they imply that the function $F(x,v,\bar v)$ is homogeneous of first degree and invariant under phase transformations acting on the v^I. Furthermore the irrelevant terms can be chosen to make $F_{x^I v^J}$ symmetric in I and J. Henceforth we restrict the function $F(x,v,\bar v)$ accordingly, which implies that it satisfies the following differential equations:

$$\begin{aligned} F_{x^I x^J} + F_{v^I \bar v^J} &= 0\,, \\ x^I F_{x^I} + v^I F_{v^I} + \bar v^I F_{\bar v^I} &= F\,, \\ v^I F_{v^I} - \bar v^I F_{\bar v^I} &= 0\,, \\ F_{x^I v^J} - F_{x^J v^I} &= 0\,. \end{aligned} \tag{5.10}$$

Moreover, it turns out that the phase transformations are part of an SU(2) group, under which the lagrangian is also invariant. Under the SU(2) transformations x^I, v^I and $\bar v^I$ transform as vectors,

$$\delta v^I = i\epsilon^3 v^I + \epsilon^+ x^I\,, \quad \delta\bar v^I = -i\epsilon^3 \bar v^I + \epsilon^- x^I\,, \quad \delta x^I = -2(\epsilon^- v^I + \epsilon^+ \bar v^I)\,, \tag{5.11}$$

and leave $x^I x^J + 2v^I\bar v^J + 2\bar v^I v^J$ invariant for all I and J. Equations (5.10) imply that the lagrangian (5.2) is SU(2) invariant up to a total derivative. The only nontrivial part of the calculation arises in proving the invariance (modulo total derivatives) of the interaction linear in the tensor field; an intermediate step is:

$$\delta\Big(F_{x^I v^J}\partial_\mu v^J - F_{x^I\bar v^J}\partial_\mu \bar v^J\Big) = \epsilon^+\,\partial_\mu F_{v^I} - \text{ h.c.}\,. \tag{5.12}$$

We stress that this SU(2) invariance, which is crucial for $N=2$ superconformal invariance, is automatic at this point. Hence we conclude that the lagrangians based on functions satisfying (5.10) encode all the $N=2$ superconformal theories of n tensor supermultiplets.

When performing the duality transformation, SU(2) variations proportional to the divergence of H^I_μ no longer vanish identically and must be cancelled by assigning a suitable SU(2) transformation to the fields y_I. This determines the SU(2) transformations of the fields v^I, w_I and their complex conjugates. The U(1) transformations with parameter ϵ^3 still act only on the v^I by a uniform change of phase. Under the remaining transformations we have

$$\delta v^I = \epsilon^+ k^{-v^I}\,, \qquad \delta w_I = \epsilon^+\, k^{-w_I}\,, \tag{5.13}$$

with

$$k^{-v^I} = x^I\,, \qquad k^{-w_I} = F_{v^I}\,. \tag{5.14}$$

One can show, always using (5.10), that these transformation rules correctly generate the SU(2) algebra. With lower indices, the vectors take a very simple form,

$$k^+_{v^I} = 0\,, \qquad k^+_{w_I} = 2\,v^I\,, \tag{5.15}$$

where we made use of (5.10). Observe that this vector depends exclusively on the coordinates v^I and is thus holomorphic. Because k^{-v^I} and k^{-w_I} do not depend on the imaginary part of w_I, the SU(2) transformations commute with the isometries associated with purely imaginary shifts of the w_I. The scale transformations also commute with these isometries, because w_I has zero scaling weight.

We have thus shown that the resulting hyper-Kähler manifold has an SU(2) isometry group. Conversely one can show directly that the superconformal invariance of the hypermultiplet theories is carried over to the tensor multiplets. This is not the general situation with regard to other invariances: the imaginary shifts of the fields w_I in the hypermultiplet description act trivially on the tensor multiplets, and symmetries that do not commute with these shifts do not induce symmetries of the tensor multiplet action. This is similar to the situation that one has when dualizing vector multiplets in three spacetime dimensions to hypermultiplets [23].

We now show that the Kähler potential (5.9) is the hyper-Kähler potential of a hyper-Kähler cone; this potential can be rewritten as

$$\chi(v,w,\bar v,\bar w) = F(x,v,\bar v) - (w_I+\bar w_I)\,x^I = 2v^I F_{v^I}(x,v,\bar v)\,, \tag{5.16}$$

and is invariant under SU(2) transformations, as one can explicitly check. Under scale transformations the coordinates v^I transform with weight 2, whereas the w_I are invariant, so that χ has weight two. To verify the homothety equation (2.1), we determine the homothetic Killing vectors (χ^{v^I},χ^{w_I}) obtained from χ and establish that they are holomorphic. The derivatives of the hyper-Kähler potential are

$$\chi_{v^I} = F_{v^I}\,, \qquad \chi_{w_I} = -x^I\,. \tag{5.17}$$

Using (5.10), we find

$$\chi^{v^I} = 2v^I\,, \qquad \chi^{w_I} = 0\,, \tag{5.18}$$

which indeed are holomorphic. Furthermore, we have the correct normalization conditions:

$$\chi^{v^I}\chi_{v^I} + \chi^{w_I}\chi_{w_I} = k^{-v^I}k^+_{v^I} + k^{-w_I}k^+_{w_I} = \chi\,, \tag{5.19}$$

confirming that χ is the hyper-Kähler potential. Finally we can read off the complex structures, by using the relation (2.6). One finds that J^3 is indeed the canonical one, i.e. $J^{3a}{}_b = i\delta^a{}_b$, where now the indices a,b run over both v^I and w_I. The holomorphic two-form, associated with J^+ is

$$\Omega = \mathrm{d}w_I \wedge \mathrm{d}v^I\,, \tag{5.20}$$

where we made use of (5.15). Evidently, imaginary shifts of w_I preserve the complex structures, so that these isometries are triholomorphic.

To illustrate the above results we present the simple example of a single tensor multiplet [24]. The corresponding hyper-Kähler cone is flat.[7] The example is based on the following homogeneous function, invariant under phase transformations [22],

$$F(x,v,\bar{v}) = r - x\left[\ln(x+r) - \frac{1}{2}\ln(4\,v\bar{v})\right], \tag{5.21}$$

where $r = \sqrt{x^2 + 4\,v\bar{v}}$. Its relevant derivatives are equal to

$$\begin{aligned} F_x &= -\ln(x+r) + \frac{1}{2}\ln(4\,v\bar{v}), & F_v &= \frac{r}{2\,v}, \\ F_{xx} &= -F_{v\bar{v}} = -\frac{1}{r}, & F_{xv} &= \frac{x}{2\,vr}. \end{aligned} \tag{5.22}$$

It is now easy to see that the lagrangian (5.2) is invariant under SU(2) transformations, with the exception of the term linear in the tensor field strength. However, rewriting this term in the form (5.3), SU(2) invariance becomes manifest [24]:

$$\begin{aligned} \mathcal{L} = &-\frac{1}{r}\Big(\partial_\mu v\,\partial^\mu\bar{v} + \frac{1}{4}(\partial_\mu x\,\partial^\mu x - H_\mu H^\mu)\Big) + \\ &+\frac{1}{2r^3}\Big(\bar{v}\,\partial_\mu v\,\partial_\nu x + v\,\partial_\mu x\,\partial_\nu\bar{v} + x\,\partial_\mu\bar{v}\,\partial_\nu v\Big)\,i\,\varepsilon^{\mu\nu\rho\sigma}\,B_{\rho\sigma}\ . \end{aligned} \tag{5.23}$$

After the duality transformation we obtain the new variable w whose real part is given by $w + \bar{w} = F_x$. Solving x in terms of $w, \bar{w}, v, \bar{v}$ gives

$$x = -2\sqrt{v\bar{v}}\,\sinh(w+\bar{w})\,. \tag{5.24}$$

The hyper-Kähler potential for the corresponding hyper-Kähler cone is equal to

$$\chi(v,w,\bar{v},\bar{w}) = F - (w+\bar{w})x = 2\sqrt{v\bar{v}}\,\cosh(w+\bar{w})\ . \tag{5.25}$$

Computing the line element, we find a flat metric

$$\begin{aligned} \mathrm{d}s^2 &= \frac{\cosh(w+\bar{w})}{\sqrt{v\bar{v}}}\Big[\frac{1}{2}\mathrm{d}v\,\mathrm{d}\bar{v} + 2\,v\bar{v}\,\mathrm{d}w\,\mathrm{d}\bar{w}\Big] + \frac{\sinh(w+\bar{w})}{\sqrt{v\bar{v}}}\Big[\bar{v}\,\mathrm{d}v\,\mathrm{d}\bar{w} + v\,\mathrm{d}w\,\mathrm{d}\bar{v}\Big] \\ &= \Big|\mathrm{d}(\mathrm{e}^{w}\sqrt{v})\Big|^2 + \Big|\mathrm{d}(\mathrm{e}^{-w}\sqrt{v})\Big|^2. \end{aligned} \tag{5.26}$$

To end this section we turn to the question of finding explicit realizations of the function F. As was shown in [16, 17], a function F that satisfies (5.1) can be represented as a contour integral:

$$F = \mathrm{Re}\oint \frac{\mathrm{d}\xi}{2\pi i\xi}\,G(\eta(\xi),\xi)\,, \tag{5.27}$$

[7] A flat space can also be obtained from a tensor lagrangian which is not conformally invariant. Namely, choose $F(x,v,\bar{v}) = v\bar{v} - 1/2x^2$, which leads to $K(v,w,\bar{v},\bar{w}) = v\bar{v} + w\bar{w}$, up to a Kähler transformation. However, the isometries associated with imaginary shifts of w do not commute with SU(2) in this case.

where

$$\eta^I(\xi) = \bar{v}^I \,\xi^{-1} + x^I - v^I \,\xi\,. \tag{5.28}$$

It follows by straightforward calculation that this form of F satisfies (5.1) and that $F_{x^I v^J}$ is symmetric in I and J. Furthermore, the conformal constraints (5.10) translate into simple constraints on the function $G(\eta(\xi), \xi)$. Firstly, the homogeneity constraint in (5.10) requires G to be homogeneous of first degree in the variables η (under the contour integral),

$$\oint \frac{\mathrm{d}\xi}{2\pi i\xi}\,\eta^I \frac{\partial G}{\partial \eta^I} = \oint \frac{\mathrm{d}\xi}{2\pi i\xi}\,G\,. \tag{5.29}$$

Secondly, the SO(2) invariance of F implies that $G(\eta(\xi), \xi)$ is only a function of $\eta(\xi)$, so there is no explicit ξ dependence. Indeed, one computes

$$\begin{aligned} 0 \;=\; (\bar{v}^I\partial_{\bar{v}^I} - v^I\partial_{v^I})F &= \oint \frac{\mathrm{d}\xi}{2\pi i\xi}\frac{\partial G}{\partial \eta^I}\Big(\frac{\bar{v}^I}{\xi} + \xi\, v^I\Big) \\ &= -\oint \frac{\mathrm{d}\xi}{2\pi i\xi}\,\frac{\partial G}{\partial \eta^I}\,\xi\,\frac{\partial \eta^I}{\partial \xi} \\ &= -\oint \frac{\mathrm{d}\xi}{2\pi i}\Big(\frac{\mathrm{d}}{\mathrm{d}\xi}G - \frac{\partial G}{\partial \xi}\Big)\,. \end{aligned} \tag{5.30}$$

The first term in the last line between brackets is zero because it is a total derivative and the contour is closed. The whole expression therefore vanishes if G is only a function of $\eta(\xi)$ and there is no explicit ξ dependence.[8] For the hyper-Kähler potential one thus obtains the expression

$$\chi(v, \bar{v}, w, \bar{w}) = \mathrm{Re}\,\left[-2v^I \oint \frac{\mathrm{d}\xi}{2\pi i}\,\frac{\partial G(\eta)}{\partial \eta^I}\right], \tag{5.31}$$

where the coordinates w_I satisfy

$$(w+\bar{w})_I = \mathrm{Re}\oint \frac{\mathrm{d}\xi}{2\pi i\xi}\,\frac{\partial G(\eta)}{\partial \eta^I}\,. \tag{5.32}$$

There are plenty of examples, the simplest one, corresponding to (5.21), is [16]

$$G(\eta) = \eta \ln \eta\,, \tag{5.33}$$

with a contour turning clockwise and anticlockwise around the two roots of $\eta(\xi) = 0$ (contributions from a contour around the origin are irrelevant as they lead to total derivatives in the action). This function is indeed homogeneous under the contour integral since terms linear in η do not contribute to F.

[8]In fact, $\partial G/\partial\xi$ should only vanish under the contour integral, but in most interesting examples, we have the stronger relation $\partial G/\partial \xi = 0$.

The duality between tensor multiplets and hypermultiplets with triholomorphic isometries bears a close relation to the **c**-map, which maps $N = 2$ supersymmetric abelian vector multiplets to hypermultiplets, again with triholomorphic isometries. The combined map[9] interchanges vector multiplets and tensor multiplets. The vector multiplet lagrangian is encoded in a holomorphic function $\mathcal{F}(v)$, where the complex vector multiplet scalars are denoted by v^I. The corresponding function F that characterizes the lagrangian of the tensor multiplets is given by

$$F(x, v, \bar{v}) = \mathrm{Re}\Big[- i\,\bar{v}^I\,\mathcal{F}_I(v) + \frac{1}{2} i\,x^I x^J\,\mathcal{F}_{IJ}(v)\Big], \tag{5.34}$$

which obviously satisfies (5.1). There is a corresponding function G (see (5.27)), evaluated with a contour around the origin:

$$G = \frac{i\mathcal{F}(\xi\eta)}{2\xi^2}\,. \tag{5.35}$$

Note that conformal invariance is not preserved by this map. It is well-known that the functions $\mathcal{F}(v)$ belong to certain equivalence classes (via electric-magnetic duality); likewise the functions $G(\eta)$ fall into certain equivalence classes. Dualizing to the corresponding hyper-Kähler space in both cases, we find that these equivalences are the same and arise from considering different sets of triholomorphic isometries on the hyper-Kähler space (see section 7).

6. Abelian quaternionic isometries

In this section we discuss the quaternion-Kähler geometry associated with the hyper-Kähler cones from the last section. These hyper-Kähler cones have n abelian triholomorphic isometries. Any $4n$ (real) dimensional hyper-Kähler cone with n such isometries can be characterized in terms of a homogeneous function F, because we can always dualize back to a description in terms of n tensor multiplets. However, there may exist equivalent but different tensor multiplet lagrangians when there are more than n triholomorphic isometries and one can find inequivalent subsets of n isometries that commute; we return to this issue in the next section. According to section 4, the n abelian triholomorphic isometries of the $4n$-dimensional hyper-Kähler cone descend to n abelian quaternionic isometries. This approach gives a complete classification of all quaternion-Kähler spaces of dimension $4(n-1)$ with n quaternionic abelian isometries as follows: such a space can be dualized by introducing n tensors, so that one is left with a configuration of n tensors and $3n-4$ scalars. In the supergravity context, one can argue that this coupling can be described by a superconformal theory of n tensor multiplets,

[9] This vector-tensor multiplet duality is actually more direct than either the **c**-map or the tensor-hypermultiplet duality as it does not involve the equations of motion.

which must thus be in the class discussed in the previous section. Therefore the classification of the quaternionic spaces with n abelian isometries must be complete.[10]

We recall (c.f. (5.16)) that the hyper-Kähler potential $\chi(w,\bar{w},v,\bar{v}) = 2v^I F_{v^I}(x,v,\bar{v})$, where the coordinates x^I are solved in terms of $w_I+\bar{w}_I, v^I$ and $\bar{v}^I$ by (5.5). The first step in the construction of the quaternion-Kähler space is to reduce to the twistor space $\mathcal{Z}$ by singling out the z coordinate as explained in section 3 (cf. (3.3)). The homothety was given in (5.18), and hence we choose

$$2v^I\partial_{v^I} \equiv \partial_z\,. \tag{6.1}$$

Since the coordinates on the twistor space have weight zero under dilatations, it is convenient to use special coordinates (compatible with (6.1)):

$$v^I = \mathrm{e}^{2z}t^I\,, \qquad \text{with } I=1,\ldots,n-1\,, \qquad v\equiv v^n = \mathrm{e}^{2z}\,. \tag{6.2}$$

The twistor space $\mathcal{Z}$ is then parametrized by $2n-1$ complex coordinates, t^I, w_I, and $w\equiv w_n$, with $I=1,\ldots,n-1$ (here and henceforth). All these coordinates have zero weight under dilatations. It is now convenient to change the coordinates x^I of the tensor multiplets in a similar way,

$$x^I = \mathrm{e}^{z+\bar{z}}q^I\,, \qquad x\equiv x^n = \mathrm{e}^{z+\bar{z}}q\,, \tag{6.3}$$

where the q^I are real and have zero scaling weight.

Because $F(x,v,\bar{v})$ is homogeneous and invariant under SO(2), it can be written in terms of a new function $\mathcal{F}$ as

$$F(x,v,\bar{v}) = \mathrm{e}^{z+\bar{z}}\mathcal{F}(q,t,\bar{t})\,, \tag{6.4}$$

which is still restricted by the first and the last equation of (5.10). We give these restrictions below. The derivatives with respect to the new coordinates can be expressed straightforwardly in terms of the old ones,

$$\begin{aligned} \partial_{x^I} &= \mathrm{e}^{-(z+\bar{z})}\partial_{q^I}\,, & \partial_{v^I} &= \mathrm{e}^{-2z}\partial_{t^I}\,,\\ \partial_x &= \mathrm{e}^{-(z+\bar{z})}\partial_q\,, & \partial_v &= \frac{1}{2}\mathrm{e}^{-2z}(\partial_z - 2t^I\partial_{t^I} - q^I\partial_{q^I} - q\,\partial_q)\,. \end{aligned} \tag{6.5}$$

It follows that the Kähler potential of the twistor space is determined by the function $\mathcal{F}$,

$$K(t,\bar{t},w+\bar{w}) = \ln\Big[\mathcal{F}(q,t,\bar{t}) - q^I(w_I+\bar{w}_I) - q(w+\bar{w})\Big], \tag{6.6}$$

where the coordinates q^I and q are determined as functions of t^I, w_I and w from

$$w_I+\bar{w}_I = \frac{\partial\mathcal{F}}{\partial q^I}\,, \qquad w+\bar{w} = \frac{\partial\mathcal{F}}{\partial q}\,. \tag{6.7}$$

[10]The quotient construction for the tensor multiplets is subtle for $n=1$ because there are not enough scalars that can be gauge-fixed; this case has been worked out in [24].

We can also write down the constraints corresponding to the first equation of (5.10).

$$
\begin{aligned}
\mathcal{F}_{q^I q^J} + \mathcal{F}_{t^I \bar{t}^J} &= 0\,, \\
\mathcal{F}_{t^I} + 2\mathcal{F}_{q^I q} - 2\mathcal{F}_{t^I \bar{t}^J}\bar{t}^J - \mathcal{F}_{t^I q^J} q^J - \mathcal{F}_{t^I q}\, q &= 0\,, \\
\frac{1}{4}\mathcal{F} - \frac{1}{4}(q^I \mathcal{F}_{q^I} - q\mathcal{F}_q) + \left(1 + \frac{1}{4}q^2\right)\mathcal{F}_{qq} + \left(t^I \bar{t}^J + \frac{1}{4}q^I q^J\right)\mathcal{F}_{q^I q^J} + & \\
+\left(t^I + \bar{t}^I + \frac{1}{2}q^I q\right)\mathcal{F}_{q^I q} &= 0\,.
\end{aligned}
\tag{6.8}
$$

The last equation of (5.10) implies

$$
\begin{aligned}
\mathcal{F}_{q^I t^J} - \mathcal{F}_{q^J t^I} &= 0\,, \\
\mathcal{F}_{t^I q} + t^J \mathcal{F}_{t^J q^I} + \frac{1}{2}q^J \mathcal{F}_{q^J q^I} + \frac{1}{2}q\mathcal{F}_{q^I q} &= 0\,.
\end{aligned}
\tag{6.9}
$$

In the previous section we discussed how the function F can be represented by a contour integral (cf. (5.27)). There is a corresponding representation for the function $\mathcal{F}$ on the twistor space $\mathcal{Z}$. Absorbing a phase factor in the contour integration variable ξ, one can straightforwardly verify the following expression,

$$
\mathcal{F}(q,t,\bar{t}) = \oint \frac{\mathrm{d}\xi}{2\pi i\xi}\, G(\eta(\xi))\,, \tag{6.10}
$$

where G is still a homogeneous function in the η's (up to terms that vanish under the contour integral). The latter are now defined by

$$
\eta^I(\xi) = \bar{t}^I\,\xi^{-1} + q^I - t^I\,\xi\,, \qquad \eta^n(\xi) = \xi^{-1} + q - \xi\,. \tag{6.11}
$$

The holomorphic hyper-Kähler cone two-form Ω was given in (5.20); in the new coordinates, it becomes

$$
\Omega = \mathrm{e}^{2z}\left[2(\mathrm{d}w + t^I \mathrm{d}w_I)\wedge \mathrm{d}z + \mathrm{d}w_I \wedge \mathrm{d}t^I\right]. \tag{6.12}
$$

From this result one can easily read off the holomorphic one-form X and the holomorphic two-form ω on the twistor space,

$$
\begin{aligned}
X &= 2(\mathrm{d}w + t^I \mathrm{d}w_I)\,, \\
\omega &= \mathrm{d}w_I \wedge \mathrm{d}t^I\,.
\end{aligned}
\tag{6.13}
$$

By computing the inverse two-form, we find Y^i, and hence (see (4.8))

$$
\zeta = 2w\,. \tag{6.14}
$$

Decomposing the holomorphic HKC two-form and its inverse in terms of the coordinates (t^I, w_I, ζ, z) one obtains the following results for the quaternion-Kähler manifold $\mathbf{Q}^{4(n-1)}$:

$$
\omega_{\alpha\beta} = \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \qquad X_\alpha(t) = (0, 2\,t^I)\,, \qquad Z^\alpha(t) = (2\,t^I, 0)\,. \tag{6.15}
$$

Observe that these quantities only depend on t^I and not on w_I; this can be understood from the presence of the triholomorphic isometries, as discussed below.

At this point one can evaluate the twistor-space metric by taking appropriate derivatives of the Kähler potential (6.6). This result can be expressed in terms of the derivatives of the function $\mathcal{F}$, along the same lines as in the beginning of section 5. For instance, we note that

$$K_{t^I} = \mathcal{F}_{t^I}\, \mathrm{e}^{-K}\,, \quad K_{w_I} = -q^I\, \mathrm{e}^{-K}\,, \quad K_\zeta = -\frac{1}{2} q\, \mathrm{e}^{-K}\,. \tag{6.16}$$

With these result we can write down the quaternion-Kähler metric by following the procedure outlined in section 4. We restrict the indices to the coordinates (t^I, w_I) and subsequently impose the gauge condition $\zeta = 0$, which implies $\partial\mathcal{F}/\partial q = 0$. This equation determines q in terms of the other coordinates, $q = q(q^I, t^I, \bar{t}^I)$, and hence the scalars of the n-th tensor multiplet are completely eliminated. The different components of the quaternion-Kähler metric are given by

$$\begin{aligned} G_{w_I\bar{w}_J} &= K_{w_I\bar{w}_J} - 4\mathrm{e}^{-2K} t^I \bar{t}^J\,, \\ G_{w_I\bar{t}^J} &= K_{w_I\bar{t}^J}\,, \\ G_{t^I\bar{t}^J} &= K_{t^I\bar{t}^J}\,. \end{aligned} \tag{6.17}$$

Likewise we evaluate the quaternion-Kähler two-form $\mathcal{Q}^+$ defined by (4.4):

$$\mathcal{Q}^+ = \mathrm{e}^{-K}\Big[\mathrm{d}w_I \wedge \Big(\mathrm{d}t^I - 2t^I K_{t^J}\,\mathrm{d}t^J\Big) - 2t^I K_{w_J}\,\mathrm{d}w_I \wedge \mathrm{d}w_J\Big]\,. \tag{6.18}$$

All the above formulae are evaluated at $\zeta = 0$.

We now discuss the abelian isometries. In terms of the coordinates introduced above, the n commuting triholomorphic isometries in the hyper-Kähler cone take the form

$$\delta w_I = i\alpha_I\,, \qquad \delta w = i\alpha\,, \qquad \delta t^I = 0\,, \qquad \delta z = 0\,, \tag{6.19}$$

where α and α_I are real, constant parameters. It then follows trivially that the twistor space has the same set of isometries and that these are holomorphic with respect to the complex structure on the twistor space. This is in accord with the general discussion at the end of section 3. We find that the moment map corresponding to the shift in w_I is simply given by $\mu_{(I)} = -K_{\bar{w}_I} = -K_{w_I}$, and the one corresponding to the shift in w is given by $\mu = -K_{\bar{w}} = -K_w$.

In the quaternion-Kähler manifold, the shifts in w_I still preserve the metric (6.17) and thus remain isometries: $\zeta \equiv 2w$ is not affected by them, and hence the gauge choice $\zeta = 0$ induces no compensating Sp(1) transformation. This is different for the shift in w, which has to be compensated by an Sp(1) transformation according to (4.29). Because the shift in the twistor space acts exclusively on ζ and not on any of the other coordinates, the corresponding isometry in the quaternion-Kähler space is directly proportional to X^α. Hence the Killing vectors associated with n the quaternionic isometries are given by

$$k_{(I)}^{w_J} = i\,\delta_J^I\,, \qquad k^\alpha = i\frac{X^\alpha}{X^\zeta}\,, \tag{6.20}$$

and their complex conjugates. In these formulae one again sets $\zeta = 0$. It follows easily that these isometries commute, since X^i depends on $w_I + \bar{w}_I$. Of course, this is all in accord with the general structure discussed in section 4. Likewise the quaternionic structure is invariant under the shifts in w_I, as can be explicitly verified from (6.18), while under the n-th isometry it rotates according to (4.32).

7. The universal hypermultiplet

In this section we discuss the universal hypermultiplet, which parametrizes a four-dimensional quaternion-Kähler manifold that appears as part of the moduli space of Calabi-Yau compactifications of type-II strings. Classically the relevant homogeneous space is [6]

$$\mathbf{Q}_{\rm UH} = \frac{\mathrm{U}(1,2)}{\mathrm{U}(1)\times \mathrm{U}(2)}\,, \tag{7.1}$$

which is the lowest dimensional case of the Wolf spaces $X(n-1)$. These spaces, which are homogeneous quaternion-Kähler manifolds of real dimension $4(n-1)$, are given by [25],

$$X(n-1) = \frac{\mathrm{U}(n-1,2)}{\mathrm{U}(n-1)\times \mathrm{U}(2)}\,. \tag{7.2}$$

The $X(n-1)$ belong to the class of Kähler spaces $G(p,q)$ of real dimension $2pq$,

$$G(p,q) = \frac{\mathrm{U}(p,q)}{\mathrm{U}(p)\times \mathrm{U}(q)}\,, \tag{7.3}$$

called grassmannian manifolds. To exhibit the distinction between the parametrization of $X(n-1)$ as a quaternion-Kähler and as a Kähler manifold, let us briefly discuss the coset representative for the $G(p,q)$ and their corrsponding Kähler potentials.

We start with the coset representative (see e.g. [26]), decomposed as follows,

$$\frac{\mathrm{U}(p,q)}{\mathrm{U}(p)\times \mathrm{U}(q)} = \begin{pmatrix} \sqrt{\mathbf{1}_p \mp XX^\dagger} & X \\ \mp X^\dagger & \sqrt{\mathbf{1}_q \mp X^\dagger X} \end{pmatrix}, \tag{7.4}$$

where X is a p-by-q complex matrix and the upper(lower) sign describes the compact(noncompact) version; in the compact case, $\mathrm{U}(p,q)$ is replaced by $\mathrm{U}(p+q)$. The pq complex parameters contained in X provide (local) coordinates on $G_{(p,q)}$.

To exhibit the Kähler structure, it is convenient to adopt inhomogenous coordinates defined by $Z = X\,[\mathbf{1}_q \mp X^\dagger X]^{-1/2}$. Then the line element takes the form,

$$\mathrm{d}s^2 = \pm\,\mathrm{Tr}\left[\frac{1}{\mathbf{1}_p \pm ZZ^\dagger}\,\mathrm{d}Z\,\frac{1}{\mathbf{1}_q \pm Z^\dagger Z}\,\mathrm{d}Z^\dagger\right]. \tag{7.5}$$

Clearly the metric is hermitean in the coordinates Z and $Z^\dagger$; it has a Kähler potential,

$$K(Z,Z^\dagger) = \mathrm{Tr}\Big[\ln[\mathbf{1}_p \pm ZZ^\dagger]\Big]\,. \tag{7.6}$$

However, as we shall see in the next subsection, the coordinates Z are not the appropriate coordinates for the quaternion-Kähler description of $X(n-1)$.

JHEP02(2001)039

Returning to the universal hypermultiplet space $\mathbf{Q}_{\rm UH}$ we mention that one finds different expressions for its Kähler potential in the literature. One is based on complex coordinates S and C, which refer to the dilaton/axion complex and the R-R fields, with

$$K_{\rm UH} = \ln(S + \bar{S} - 2C\bar{C})\,. \tag{7.7}$$

Another one, based on coordinates u and v and Kähler potential

$$K_{\rm UH} = \ln(1 - u\bar{u} - v\bar{v})\,, \tag{7.8}$$

corresponds directly to the parametrization (7.6). The two potentials are related by a holomorphic coordinate transformation

$$S = \frac{1-u}{1+u}\,, \qquad C = \frac{v}{1+u}\,, \tag{7.9}$$

and a Kähler transformation $\ln[(1+u)/\sqrt{2}] + h.c.$.

We now describe $\mathbf{Q}_{\rm UH}$ starting from an appropriate hyper-Kähler cone, using the techniques of this paper. In the first subsection, we construct this hyper-Kähler cone and its corresponding twistor space for all $X(n-1)$, and specialize at the end to $n = 2$. In a second subsection, we rederive the same geometries using the Legendre transform method and the contour integral formalism of section 5. The hyper-Kähler geometry associated to the Wolf spaces has also been discussed in the mathematics literature [27].

7.1 Quotient construction of $X(n-1)$

It is well known [28, 29] that the Wolf spaces $X(n-1)$ can be obtained by performing a hyper-Kähler quotient of a flat space of complex dimension $2n+2$ followed by an $N = 2$ superconformal quotient (the two quotients can be performed in opposite order: first the $N = 2$ superconformal quotient followed by a quaternionic one [29]). In this section we first perform the hyper-Kähler quotient and obtain a $4n$-dimensional hyper-Kähler cone and the corresponding twistor space $\mathcal{Z}$. Subsequently we obtain the quaternion-Kähler space via the procedure outlined in section 4. Thus we start by considering $\mathbf{C}^{2n+2}$, with a pseudo-riemannian metric with $n+1$ pairs of complex coordinates denoted by (z_+^I, z_{-I}). We can distinguish a number of obvious symmetry groups that act linearly on these coordinates. First, z_+ and z_- transform in the (inequivalent) conjugate fundamental representations of $\mathrm{U}(n-1,2)$ (or its compact version) so that their product is invariant. Hence, when z_+ transforms as $z_+^I \to U^I{}_J\, z_+^J$, then z_- transforms according to $z_{-I} \to (U^{-1})^J{}_I\, z_{-J}$. The noncompact versions of $\mathrm{U}(n+1)$ satisfy $(U^{-1})^I{}_J = \eta^{I\bar{I}}\,\bar{U}^{\bar{J}}{}_{\bar{I}}\,\eta_{J\bar{J}}$, where $\eta_{I\bar{J}}$ is a diagonal matrix with entries ± 1, and $\eta^{I\bar{J}}$ is its inverse. Furthermore, z_+ and $\eta\bar{z}_-$ transform as a doublet under $\mathrm{SU}(2)$; this is the $\mathrm{Sp}(1)$ that rotates the complex structures (2.9).

The above assignments under $\mathrm{U}(n-1,2)\times\mathrm{SU}(2)$ are characteristic for the coset-space structure of $X(n-1)$, but the holomorphic assignements are different. To be specific, the matrix X in the coset representative (cf. (7.4) with $p=n-1$ and $q=2$) is related to $(z_+^I, \eta^{I\bar{J}}\bar{z}_{-J})$.

The flat space $\mathbf{C}^{2n+2}$ is obviously a hyper-Kähler cone, and its hyper-Kähler potential is

$$\chi_{(2n+2)} = \eta_{I\bar{J}}\, z_+^I \bar{z}_+^J + \eta^{I\bar{J}}\, z_{-I}\bar{z}_{-J}\,. \tag{7.10}$$

We assume that the last two eigenvalues of $\eta_{I\bar{J}}$ are positive. For the noncompact spaces the remaining entries are equal to -1, while for the compact spaces η is the unit matrix. Furthermore, the $\mathrm{U}(n-1,2)$ invariant holomorphic two-form is given by

$$\Omega = \mathrm{d}z_+^I \wedge \mathrm{d}z_{-I}\,. \tag{7.11}$$

The isometry we quotient $\mathbf{C}^{2n+2}$ by is the $\mathrm{U}(1)$ subgroup of $\mathrm{U}(n-1,2)$ which acts on z_+ and z_- with opposite phase. The three moment maps associated with this isometry are

$$\mu^+ = -i z_+^I\, z_{-I}\,, \qquad \mu^- = \bar{\mu}^+\,, \qquad \mu^3 = -\eta_{I\bar{J}}\, z_+^I \bar{z}_+^J + \eta^{I\bar{J}}\, z_{-I}\bar{z}_{-J}\,. \tag{7.12}$$

The hyper-Kähler quotient [22, 17] is performed by gauging the isometry, integrating out the corresponding connection and setting the moment maps to zero. Note that this hyper-Kähler quotient is consistent with $\mathrm{U}(n-1,2)\times\mathrm{SU}(2)$.

One can now proceed in two equivalent ways. One is to introduce gauge-invariant inhomogeneous coordinates z_+^I/z_+^{n+1} and $z_{-I}z_+^{n+1}$. Because of $\mathrm{U}(1)$ gauge invariance the phase of z_+^{n+1} drops out while the vanishing of the moment maps implies that $|z_+^{n+1}|$ and $z_{-n+1}z_+^{n+1}$ are constrained in terms of the remaining (inhomogeneous) coordinates. We are thus left with only n coordinates z_+^a and z_{-a} with $a=1,\ldots,n$. The resulting space is still a hyper-Kähler cone whose potential (now in terms of the inhomogeneous coordinates) equals

$$\chi_{(2n)} = 2\,\chi_+\,\chi_-\,, \tag{7.13}$$

where

$$\chi_+ = \sqrt{\eta_{I\bar{J}}\, z_+^I \bar{z}_+^J}\,, \qquad \chi_- = \sqrt{\eta^{I\bar{J}}\, z_{-I}\bar{z}_{-J}}\,, \tag{7.14}$$

and

$$z_+^{n+1} = 1\,, \qquad z_{-n+1} = -z_+^a\, z_{-a}\,. \tag{7.15}$$

The dilatations act on z_+ and z_- with scaling weights 0 and 2, respectively. Furthermore the holomorphic two-form (7.11) takes the form

$$\Omega = \mathrm{d}z_+^a \wedge \mathrm{d}z_{-a} \tag{7.16}$$

on the quotient.

The same results follow using the $N=1$ superspace formalism to gauge the U(1) isometry. The hyper-Kähler potential (7.10) is gauged to

$$\hat{\chi}_{(2n+2)} = \mathrm{e}^{V} \eta_{I\bar{J}}\, z_+^I \bar{z}_+^J + \mathrm{e}^{-V} \eta^{I\bar{J}}\, z_{-I} \bar{z}_{-J}\,. \tag{7.17}$$

The (anti)holomorphic moment maps $\mu^{\pm}$ are unchanged, and μ^3 becomes

$$\hat{\mu}^3 = -\mathrm{e}^{V} \eta_{I\bar{J}}\, z_+^I \bar{z}_+^J + \mathrm{e}^{-V} \eta^{I\bar{J}}\, z_{-I} \bar{z}_{-J}\,. \tag{7.18}$$

Now we may solve $\hat{\mu}^3 = 0$ for V and substitute back into (7.17); the geometric meaning of this procedure is explained in [17]. In $N=1$ superspace, the gauge group is complexified and hence we may choose the gauge $z_+^{n+1} = 1$, while z_{-n+1} is again determined by the holomorphic momentum map constraint. This is all derived from $N=2$ superspace in appendix B.

Finally we note that the SU(2) transformations that rotate the complex structures (2.9) of the quotient take the form

$$\begin{aligned} \delta z_+^a &= \epsilon^+ \frac{\chi_+}{\chi_-} \left[\eta^{a\bar{b}} + z_+^a\, \bar{z}_+^b\right] \bar{z}_{-b}\,, \\ \delta z_{-a} &= -\epsilon^+ \left[\frac{\chi_-}{\chi_+}\, \eta_{a\bar{b}} + \frac{\chi_+}{\chi_-}\, z_{-a}\, \bar{z}_{-b}\right] \bar{z}_+^b\,. \end{aligned} \tag{7.19}$$

The SU(2) invariance of the hyper-Kähler potential can explicitly be checked using

$$\delta_{\epsilon^+} \chi_{\pm} = \pm \frac{\epsilon^+\, \chi_+^2\, \bar{z}_+^a \bar{z}_{-a}}{2\chi_{\mp}}\,. \tag{7.20}$$

A similar analysis can be done for the group of SU$(n-1,2)$ triholomorphic isometries. For finite transformations these take the form,

$$\begin{aligned} z_+^a &\to \frac{U^a{}_{n+1} + U^a{}_b\, z_+^b}{U^{n+1}{}_{n+1} + U^{n+1}{}_c\, z_+^c}\,, \\ z_{-I} &\to \left[(U^{-1})^J{}_I\, z_{-J}\right] \left[U^{n+1}{}_{n+1} + U^{n+1}{}_a\, z_+^a\right]. \end{aligned} \tag{7.21}$$

The factors $\chi_{\pm}$ of the hyper-Kähler potential transform as

$$\chi_{\pm} \to |U^{n+1}{}_{n+1} + U^{n+1}{}_a\, z_+^a|^{\mp 1}\, \chi_{\pm}\,, \tag{7.22}$$

so that χ is indeed invariant.

Now we descend to the twistor space, decomposing the coordinates as

$$z_{-n} = \mathrm{e}^{2z}\,, \qquad z_{-i} = \mathrm{e}^{2z}\, u_i \qquad (i = 1, \ldots, n-1)\,. \tag{7.23}$$

In terms of these coordinates the holomorphic two-form is

$$\Omega = 2\, \mathrm{e}^{2z} \left(\mathrm{d}z_+^n + u_i\, \mathrm{d}z_+^i\right) \wedge \mathrm{d}z + \mathrm{e}^{2z}\, \mathrm{d}z_+^i \wedge \mathrm{d}u_i\,. \tag{7.24}$$

From this we can read off the holomorpic one-form on the twistor space,

$$X = 2\,\mathrm{d}z_+^n + 2\,u_i\,\mathrm{d}z_+^i\,, \tag{7.25}$$

which shows that the coordinate ζ is given by $2z_+^n$. Hence we identify

$$2z_+^n = \zeta\,, \qquad z_+^i = v^i\,, \tag{7.26}$$

so that the holomorphic two-form is

$$\Omega = \mathrm{e}^{2z}\Big[(\mathrm{d}\zeta + 2u_i\,\mathrm{d}v^i)\wedge \mathrm{d}z + \mathrm{d}v^i \wedge \mathrm{d}u_i\Big]\,. \tag{7.27}$$

The Kähler potential of the twistor space is given by

$$K(u,v,\zeta,\bar u,\bar v,\bar\zeta) = \ln\left[2\,\chi_+(v,\zeta,\bar v,\bar\zeta)\;\chi_-(u,v,\zeta,\bar u,\bar v,\bar\zeta)\right]\,, \tag{7.28}$$

with

$$\begin{aligned}\chi_+(v,\zeta,\bar v,\bar\zeta) &= \sqrt{1+\frac{1}{4}\zeta\bar\zeta + \eta_{i\bar\jmath}\,v^i\bar v^j}\,,\\ \chi_-(u,v,\zeta,\bar u,\bar v,\bar\zeta) &= \sqrt{1+\left|\frac{1}{2}\zeta + u_i v^i\right|^2 + \eta^{i\bar\jmath}\,u_i\bar u_j}\,.\end{aligned} \tag{7.29}$$

The action of the triholomorphic $\mathrm{SU}(n-1,2)$ isometries follows straightforwardly from (7.21) and the definitions of the new coordinates u, v, ζ. All of this is in accord with the analysis presented in section 3.

We now have all the ingredients to construct the metric of the quaternionic spaces $X(n-1)$. To avoid lengthy formulae, we return to the case $n=2$, where we can drop the indices i, j. We simultaneously treat both the compact and the noncompact case by allowing $\eta = \pm 1$. Imposing the gauge condition $\zeta = 0$, the twistor space Kähler potential becomes

$$K(u,v,0,\bar u,\bar v,0) = \frac{1}{2}\ln\Big[1 \pm u\bar u(1\pm v\bar v)\Big] + \frac{1}{2}\ln\Big[1\pm v\bar v\Big] + \ln 2\,. \tag{7.30}$$

Using (7.25), the metric then follows from (4.17):

$$\begin{aligned}G_{u\bar u} &= \pm\frac{1\pm v\bar v}{2\,[1\pm u\bar u(1\pm v\bar v)]^2}\,,\\ G_{u\bar v} &= \frac{\bar u v}{2\,[1\pm u\bar u(1\pm v\bar v)]^2}\,,\\ G_{v\bar v} &= \pm\frac{1\pm u\bar u(1\pm v\bar v)^2}{2\,[1\pm u\bar u(1\pm v\bar v)]^2[1\pm v\bar v]^2}\,.\end{aligned} \tag{7.31}$$

Similarly, the quaternion-Kähler two-form $\mathcal{Q}^+$ follows from (4.19),

$$\mathcal{Q}^+ = \frac{\mathrm{d}v\wedge\mathrm{d}u}{2\,[1\pm u\bar u(1\pm v\bar v)]^{3/2}[1\pm v\bar v]^{1/2}}\,. \tag{7.32}$$

Although the metric is obviously hermitean, it is not manifestly Kähler. We already noted that the holomorphic assignments are different for the Kähler and quaternion-Kähler formulations, so there must be a *nonholomorphic* coordinate transformation to a coordinate system in which the metric is manifestly Kähler: $G_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} K_{\rm UH}$ for some Kähler potential $K_{\rm UH}$. It is not difficult to find this coordinate transformation,

$$u' = u(1 \pm v\bar{v}) \,, \qquad v' = \bar{v} \,, \tag{7.33}$$

and the corresponding Kähler potential takes the standard form,

$$K_{\rm UH} = \frac{1}{2} \ln(1 \pm u'\bar{u}' \pm v'\bar{v}') \,, \tag{7.34}$$

in accordance with (7.6). Note, however, that the quaternion-Kähler two-form $\mathcal{Q}^+$ is no longer a $(2,0)$-form in these coordinates.

7.2 Tensor multiplet description and dualities

In this subsection, we discuss how the universal hypermultiplet can be obtained from tensor multiplets via the Legendre transform method of sections 5 and 6. We seek a function F that satisfies the constraints (5.10), and yields the correct hyper-Kähler potential for the hyper-Kähler cone related to the universal hypermultiplet. We first treat the whole class of spaces $X(n-1)$. As explained at the end of section 5, the function F can be represented as a contour integral,

$$F(x^I, v^I, \bar{v}^I) = \mathrm{Re} \oint \frac{\mathrm{d}\xi}{2\pi i \xi} \, G(\eta(\xi)) \,, \tag{7.35}$$

where

$$\eta^I(\xi) = \bar{v}^I \, \xi^{-1} + x^I - v^I \, \xi \,, \tag{7.36}$$

and G is homogeneous of first degree under the contour integral.

For a free hypermultiplet, the function G is given by (5.33) and integration over a suitable contour yields the function F given in (5.21). Introducing an arbitrary sign factor σ in front of the functions F and G, such as to allow for a pseudo-riemannian metric, the identification of the coordinates is given by

$$z_\pm = \mathrm{e}^{\pm w} \sqrt{v} \,, \qquad x = -2\sigma \sqrt{v\bar{v}} \, \sinh(w + \bar{w}) \,. \tag{7.37}$$

We now consider $n+1$ free hypermultiplets, so that the function F is

$$F_{(\rm 2n+2)} = \sum_{I=1}^{n+1} \sigma_I F^I \,, \tag{7.38}$$

where F^I is the function (5.21) for the $I'th$ tensor multiplet:

$$F^I \equiv F_2(x^I, v^I, \bar{v}^I) = r^I - x^I \left[\ln(x^I + r^I) - \frac{1}{2} \ln(4\, v^I \bar{v}^I) \right] , \tag{7.39}$$

where $r^I = \sqrt{(x^I)^2 + 4v^I\bar{v}^I}$. This describes $\mathbf{C}^{2n+2}$; the $\sigma_I = \pm 1$ are the signature factors introduced before (in the previous subsection, they are $\eta_{I\bar{I}}$). The signs associated with $I = n, n+1$ are again taken positive. Note that the F^I are even functions under $x \to -x, v \to -v$, modulo terms which do not contribute to the action. Indeed, the second derivatives of F_2 are symmetric with respect to this uniform sign change (see (5.22)) and the general discussion in appendix A). The moment maps (7.12) follow now by direct substitution of (7.37) and are given by [17]

$$\mu^+ = -i \sum_I v^I \,, \qquad \mu^3 = \sum_I x^I \,. \tag{7.40}$$

There is no U(1) gauge symmetry acting on the tensor multiplet, so the hyper-Kähler quotient amounts to imposing the three constraints $\mu^\pm = \mu^3 = 0$. This leads to the elimination of x^{n+1} and v^{n+1}, and we obtain

$$F_{(2\mathrm{n})} = \sum_{a=1}^{n} \sigma_a F^a + F_2\Big(\sum_a x^a, \sum_a v^a, \sum_a \bar{v}^a\Big) \,. \tag{7.41}$$

For $n = 2$, we thus find

$$F_{\mathrm{UH}} = \sigma F_2(x^1, v^1, \bar{v}^1) + F_2(x^2, v^2, \bar{v}^2) + F_2(x^1 + x^2, v^1 + v^2, \bar{v}^1 + \bar{v}^2) \,. \tag{7.42}$$

After performing the Legendre transform, we find the eight-dimensional hyper-Kähler cone whose corresponding quaternion-Kähler space is equal to $X(1)$ or CP^2 for $\sigma = -1$ and $\sigma = 1$, respectively.

The function F_{UH} can be represented as a contour integral of[11]

$$G_{\mathrm{UH}}(\eta^1, \eta^2) = \sigma\eta^1 \ln \eta^1 + \eta^2 \ln \eta^2 + (\eta^1 + \eta^2) \ln(\eta^1 + \eta^2) \,. \tag{7.43}$$

As mentioned in the beginning of section 6, there are actually different contour integral representations for the same hyper-Kähler space. This can be understood as follows: On any hyper-Kähler manifold, a hypermultiplet can be dualized to a tensor multiplet whenever the manifold has a triholomorphic isometry. When the manifold has n *commuting* triholomorphic isometries, one can dualize n hypermultiplets to tensor multiplets. It may happen that the manifold has a non-abelian triholomorphic isometry group which contains inequivalent sets of n commuting isometries. Dualizing with respect to the different sets gives rise to different tensor multiplet actions. This situation arises for the universal hypermultiplet. As discussed in the previous subsection, the hyper-Kähler cone above $\mathbf{Q}_{\mathrm{UH}}$ has an $\mathrm{SU}(1,2)$ group of triholomorphic isometries. Because this group is noncompact, there are (at least) three

[11]The signs in this equation are somewhat symbolic, as they only make sense after the orientation of the contour is determined, and as each of the three terms are integrated along different contours. Note that whereas F_{UH} is an even function as discussed above, G_{UH} appears to be odd; this apparent discrepancy is absorbed by a change in the orientation of the contour.

inequivalent sets of commuting pairs of generators. The first set has two compact abelian isometries; we can choose the generators

$$T_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \qquad T_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{7.44}$$

The second set has one compact and one noncompact (nilpotent) isometry with generators

$$T_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \qquad T_2' = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{7.45}$$

and the third set has two noncompact isometries, nilpotent of order three and two, respectively, with generators

$$T_1' = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \qquad T_2' = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{7.46}$$

As explained in appendix B, the first set gives rise to (7.43), the second set gives

$$G_{\mathrm{UH}} = \eta^1 \ln \frac{\eta^1}{\eta^2}, \tag{7.47}$$

whereas the third set gives

$$G_{\mathrm{UH}} = \frac{(\eta^1)^2}{\eta^2}. \tag{7.48}$$

Clearly these functions are holomorphic and homogeneous of first degree, and hence correspond to hyper-Kähler cones. The last form also appeared in [30].

The functions $F(x, v, \bar{v})$ associated with these expressions for $G(\eta)$ take a rather complicated form; the functions $G(\eta)$ are the most concise way to encode the structure of the tensor lagrangians and the corresponding hyper-Kähler cones. This suggests that we should try to understand the physics in directly terms of $G(\eta)$, rather than in terms of the corresponding HKC or quaternion-Kähler space.

8. Discussion and conclusion

In this paper, we have explored the relation between hyper-Kähler cones and quaternion-Kähler geometries. This relationship is one-to-one: every hyper-Kähler cone has an associated quaternion-Kähler manifold and vice versa. From the superconformal multiplet calculus in supergravity [9] it was known how to associate a quaternion-Kähler space with a hyper-Kähler cone [11] by the $N = 2$ superconformal quotient. Here we have found explicit convenient coordinates on the quaternion-Kähler manifold by making appropriate gauge choices; along the way, we constructed the explicit Kähler potential on the twistor space of the quaternion-Kähler space. Furthermore, we have used the relation to tensor multiplets to classify $4(n-1)$-dimensional quaternion-Kähler geometries with n commuting quaternionic isometries. In principle one can also find the hyper-Kähler cone corresponding to any quaternion-Kähler manifold; although no uniform and explicit description has been given in the supergravity context, this has been shown in the mathematics literature [12, 13].

Hyper-Kähler cones and quaternion-Kähler spaces have many applications in physics. Yang-Mills instanton moduli spaces in four Euclidean dimensions have this geometrical structure. The conformal symmetry of the four-dimensional spacetime is carried over to the moduli space of the collective coordinates, so that the size of the instanton is the cone variable. In particular, the one-instanton moduli spaces corresponding to the simple gauge groups are cones over the (compact) Wolf spaces [31]; the specific spaces $X(n-1)$ discussed in section 7 appear as moduli spaces of a single $\mathrm{SU}(n+1)$ instanton. An explicit representation has been written down for the hyper-Kähler potential χ associated with the instanton moduli spaces [32],

$$\chi = \int \mathrm{d}^4x \, x^2 \, \mathrm{Tr} \, F_{\mu\nu}^2 \,, \tag{8.1}$$

which is a function of the collective coordinates of the instanton solution. It would be interesting to determine the eight-dimensional quaternion-Kähler geometry for the centered two-instanton solution in $\mathrm{SU}(2)$.

The moduli space of a Calabi-Yau compactification of type-II string theory also has a quaternion-Kähler sector which contains the dilaton. Here, few results are known for the full quantum moduli space. At the perturbative level [7], the space has a large number of continuous abelian isometries associated with the axion and the R-R fields. Beyond the perturbative level [8], one expects that there are no continuous isometries and only a number of discrete isometries may survive (much as the θ-angle in quantum-chromodynamics cannot be shifted by an arbitrary constant because of instanton corrections).

The following analogy [34] suggests that our results may lead to a determination of the quantum corrections in moduli spaces parametrized by hypermultiplets: The centered moduli space of two $\mathrm{SU}(2)$ monopoles [1], or equivalently, the Coulomb branch of three-dimensional $N = 4$ $\mathrm{SU}(2)$ Yang-Mills theory [35], is hyper-Kähler. The asymptotic (or perturbative) moduli space has a triholomorphic isometry, and is easily described in terms of an $N = 4$ vector multiplet, which is equivalent to a tensor multiplet (an $\mathrm{O}(2)$ multiplet in the terminology of appendix B, see (B.12)) in three dimensions. Nonperturbative corrections break this isometry. In the contour integral representation discussed in sections 5 and 7 and appendix B, the integrand $G(\eta(\xi))$ (which is the projective superspace lagrangian (B.5) of appendix B) is essentially unchanged; rather, the multiplet changes to an O(4) multiplet (B.13) [39]. A remarkable feature of this mechanism is that the O(4) description automatically incorporates all nonperturbative corrections with no adjustable parameters. The rigidity of the conformal structure of hyper-Kähler cones leads us to speculate that a similar miracle may occur for the Calabi-Yau moduli spaces.

Acknowledgments

We are happy to acknowledge useful discussions with N. Berkovits, K. Galicki, C. Lebrun, and W. Siegel. Some of this work was done with support from NSF grant No. PHY-9722101. B.d.W. thanks the Aspen Center for Physics for its hospitality during the workshop "String Dualities and their Applications". S.V. thanks the Erwin Schrödinger Institute for hospitality during the workshop "Holonomy Groups and Differential Geometry".

A. Superconformal constraints on $F(x,v,\bar v)$

In the text we argued that conformal invariance of the lagrangian (5.2) requires that $F_{x^Ix^J}$, and $F_{x^Iv^J}$ be homogeneous of degree -1 and covariant under phase transformation of the coordinates v^I. Hence we expect that $N=2$ superconformal invariance requires (5.1) and the following four identities (and their complex conjugates):

$$\begin{aligned}
x^K F_{x^Ix^Jx^K} + v^K F_{x^Ix^Jv^K} + \bar v^K F_{x^Ix^J\bar v^K} &= -F_{x^Ix^J}\,,\\
x^K F_{x^Iv^Jx^K} + v^K F_{x^Iv^Jv^K} + \bar v^K F_{x^Iv^J\bar v^K} &= -F_{x^Iv^J}\,,\\
v^K F_{x^Ix^Jv^K} - \bar v^K F_{x^Ix^J\bar v^K} &= 0\,,\\
v^K F_{x^Iv^Jv^K} - \bar v^K F_{x^Iv^J\bar v^K} &= -F_{x^Iv^J}\,.
\end{aligned} \tag{A.1}$$

These constraints imply that certain derivatives of the function $F(x,v,\bar v)$ are homogeneous functions of degree -1 and covariant under phase transformations of the coordinates v^I and $\bar v^I$.

We now derive the consequences of (A.1). We use (5.1) throughout this appendix. The first two equations (A.1) imply that

$$x^I F_{x^Ix^J} + v^I F_{v^Ix^J} + \bar v^I F_{\bar v^Ix^J} = c_J\,, \tag{A.2}$$

with c_J some real integration constants. Integrating once more shows that F can be decomposed according to

$$F(x,v,\bar v) = F'(x,v,\bar v) + c_J x^J\,\ln c_I x^I + f(v,\bar v)\,, \tag{A.3}$$

where F' is a homogeneous function of first degree and $f(v,\bar v)$ is any function independent of x.

Along similar lines one establishes that

$$x^I F_{x^I} + v^I F_{v^I} + \bar{v}^I F_{\bar{v}^I} - F = g(v) + h(\bar{v}, x)\,, \tag{A.4}$$

where $g(v)$ and $h(\bar{v}, x)$ are again some unknown functions. From the fact that the right-hand side of this equation must be real we deduce that $h(\bar{v}, x)$ equals the sum of $\bar{g}(\bar{v})$ and some function of x^I. Combining all information we thus find that $F(x, v, \bar{v})$ can be written as follows,

$$F(x, v, \bar{v}) = F''(x, v, \bar{v}) + \frac{1}{2} c_J x^J \,\ln |c_I v^I|^2 + G(v) + \bar{G}(\bar{v})\,, \tag{A.5}$$

where $F''(x, v, \bar{v})$ is a homogeneous function of first degree (related to F') and satisfies (5.1). Observe that the lagrangian (5.2) for the tensor multiplets only receives contributions from F'' and not from the other terms on the right-hand side, so that those can be dropped.[12]

We now analyze the last two equations of (A.1). Because all terms other than F'' already satisfy these equations, we may restrict our attention to the homogeneous function and replace F by F''. We then prove that

$$v^I F''_{v^I x^J} - \bar{v}^I F''_{\bar{v}^I x^J} = i\tilde{c}_J\,, \tag{A.6}$$

with the $\tilde{c}_J$ some real integration constants. Furthermore, using (5.1) we find

$$\frac{\partial}{\partial v^J} \frac{\partial}{\partial \bar{v}^K} \Big(v^I F''_{v^I} - \bar{v}^I F''_{\bar{v}^I} \Big) = 0\,. \tag{A.7}$$

These two results lead to the following decomposition of the function F,

$$F(x, v, \bar{v}) = F_{\rm inv}(x, v, \bar{v}) + \frac{1}{2} c_J x^J \ln |c_I v^I|^2 + \frac{1}{2} i\tilde{c}_J x^J \ln(\tilde{c}_I v^I / \tilde{c}_K \bar{v}^K) + \; G(v) + \bar{G}(\bar{v})\,, \tag{A.8}$$

where $F_{\rm inv}(x, v, \bar{v})$ is a homogeneous function of first degree which is invariant under phase transformations of the v^I and satisfies (5.1), and up to total derivatives, determines the action.

We can further restrict $F_{\rm inv}$ such that $F_{x^I v^J}$ is symmetric in I and J, as claimed in (5.10), by using the freedom to shift $F_{\rm inv}$ by terms of the form $x^I(f_I(v) + \bar{f}_I(\bar{v}))$, which do not contribute to the action. Here the $f_I(v)$ are arbitrary homogeneous and holomorphic functions of degree zero. Observe that, using (5.1), the antisymmetric part

$$a_{IJ} \equiv F_{x^I v^J} - F_{x^J v^I}\,, \tag{A.9}$$

is independent of x and $\bar{v}$, and hence is purely holomorphic in v. Moreover, the two-form a is closed, and is therefore locally exact as a function of v, i.e. $a_{IJ} = \partial_I a_J(v) - \partial_J a_I(v)$. It is then clear that if we redefine $F_{\rm inv}$ by choosing $f_I = a_I$, the new $F_{x^I v^J}$ is symmetric.

[12] Equivalently, these terms are total derivatives in $N = 1$ superspace.

All the remaining terms in the function F in (A.8) give rise only to total derivatives in the lagrangian (5.2). Therefore they can be ignored, and we restrict $F(x, v, \bar{v})$ to be homogeneous of first degree, U(1) invariant, with a symmetric $F_{x^I v^J}$ and subject to (5.1). The results lead to (5.10).

B. Projective superspace and tensor multiplet dualities

In this appendix, we review the projective superspace formalism for $N = 2$ supersymmetry [36] and use it prove the equivalence between the three different contour integral representations of the universal hypermultiplet given by (7.43), (7.47) and (7.48).

B.1 Projective superspace

This subsection is taken essentially verbatim from [36].

The algebra of $N = 2$ supercovariant derivatives in four dimensions is

$$\{D_{i\alpha}, D_{j\beta}\} = 0\,, \qquad \{D_{i\alpha}, \bar{D}^j_{\dot\beta}\} = i\delta_i^j \partial_{\alpha\dot\beta}\,, \tag{B.1}$$

where $i, j = 1, 2$ are SU(2) isospin indices and $\alpha, \dot\beta = 1, 2$ are Lorentz spinor indices. We define an abelian subspace of $N = 2$ superspace parameterized by a complex projective coordinate ξ and spanned by the supercovariant derivatives

$$\nabla_\alpha(\xi) = D_{1\alpha} + \xi D_{2\alpha}\,, \qquad \bar\nabla_{\dot\alpha}(\xi) = \bar{D}^2_{\dot\alpha} - \xi \bar{D}^1_{\dot\alpha}\,. \tag{B.2}$$

For notational simplicity we write $D_{1\alpha} = D_\alpha, D_{2\alpha} = Q_\alpha$. The conjugate of any object is constructed in this subspace by composing the antipodal map on the Riemann sphere with hermitean conjugation $\xi^* \to -1/\xi$ and multiplying by an appropriate factor. For example,

$$\bar\nabla_{\dot\alpha}(\xi) = (-\xi)\,(\nabla_\alpha)^*\left(-\frac{1}{\xi}\right) = (-\xi)\left(\bar{D}^1_{\dot\alpha} + \left(-\frac{1}{\xi}\right)\bar{D}^2_{\dot\alpha}\right)\,. \tag{B.3}$$

Throughout this appendix, all conjugates of fields and operators in projective superspace are defined in this sense.

Projective superfields living in this projective superspace obey the constraint

$$\nabla_\alpha \Upsilon = 0 = \bar\nabla_{\dot\alpha}\Upsilon\,, \tag{B.4}$$

and the restricted measure for integrating lagrangians on this subspace can be constructed from any differential operators linearly independent of ∇ and $\bar\nabla$. A convenient choice is the usual $N = 1$ measure

$$S = \oint_C \frac{\mathrm{d}\xi}{2\pi i\xi}\,\mathrm{d}^4x\; D^2\bar{D}^2 G(\Upsilon, \bar\Upsilon, \xi)\,, \tag{B.5}$$

where C is a contour in the ξ-plane that generically depends on G. The constraints (B.4) guarantee that S is $N = 2$ supersymmetric; they can be rewritten as

$$D_\alpha \Upsilon = -\xi Q_\alpha \Upsilon \,, \qquad \bar{Q}_{\dot\alpha} \Upsilon = \xi \bar{D}_{\dot\alpha} \Upsilon \,. \tag{B.6}$$

Projective superfields can be classified [37] as: i) O(k) multiplets, ii) rational multiplets, iii) analytic multiplets. We focus on O(k) multiplets, which are polynomials in ξ with powers ranging from 0 to k, and on analytic multiplets, which are analytic in some region of the Riemann sphere.

For even $k = 2p$ we impose a reality condition with respect to the conjugation defined above (see (B.3)). We use η to denote a real finite order superfield

$$\eta^{(2p)}(\xi) \equiv \frac{1}{\xi^p} \sum_{n=1}^{2p} \eta_n^{(2p)} \xi^n \,, \qquad \eta^{(2p)} = \bar{\eta}^{(2p)} \,. \tag{B.7}$$

This reality condition relates different coefficients in the ξ-expansion of η

$$\eta_{2p-n} = (-)^{p-n} \bar{\eta}_n \,. \tag{B.8}$$

There are various types of analytic multiplets. The *polar* (arctic and antarctic) multiplets are analytic around the north and south poles of the Riemann sphere, respectively:

$$\Upsilon = \sum_{n=0}^{\infty} \Upsilon_n \xi^n \,, \qquad \bar{\Upsilon} = \sum_{n=0}^{\infty} \bar{\Upsilon}_n \left(-\frac{1}{\xi}\right)^n \,. \tag{B.9}$$

The antarctic multiplet is conjugate to the arctic.

Similarly, the real *tropical* multiplet is the limit $p \to \infty$ of the real O($2p$) multiplet $\eta^{(2p)}$. It is analytic away from the polar regions, and can be regarded as a sum of a part regular at the north pole and a part regular at the south pole:

$$\mathbf{V}(\xi) = \sum_{n=-\infty}^{+\infty} \mathbf{V}_n \xi^n \,, \qquad \mathbf{V}_{-n} = (-)^n \bar{\mathbf{V}}_n \,. \tag{B.10}$$

The constraints (B.6) relate the different ξ-coefficient superfields

$$D_\alpha \Upsilon_{n+1} = -Q_\alpha \Upsilon_n \,, \qquad \bar{D}_{\dot\alpha} \Upsilon_n = \bar{Q}_{\dot\alpha} \Upsilon_{n+1} \,. \tag{B.11}$$

For any real O($2p$) multiplet these constraints are compatible with the reality condition (B.7). They also determine what type of $N = 1$ superfields the ξ-coefficients are.

Explicitly, for the O(2) multiplet,

$$\eta^{(2)} = \frac{\bar{v}}{\xi} + x - v\,\xi \,, \tag{B.12}$$

where v obeys $\bar{D}_{\dot{\alpha}} v = Q_\alpha v = 0$ and hence projects to an $N = 1$ chiral superfield, and x is real and obeys $\bar{D}^2 x = Q^2 x = 0$, and hence projects to an $N = 1$ real linear superfield. This is precisely the $N = 2$ tensor multiplet. Rigidly $N = 2$ superconformal actions for O(2) multiplets are given by (B.5),[13] where η has conformal weight two and G is homogeneous of first degree with no explicit ξ-dependence, as discussed in section 5.

Similarly, for the O(4) multiplet,

$$\eta^{(4)} = \frac{\bar{v}}{\xi^2} + \frac{\bar{s}}{\xi} + y - s\,\xi + v\,\xi^2 \,, \tag{B.13}$$

where v is constrained as for the O(2) case, s is a complex linear superfield obeying $\bar{D}^2 s = Q^2 s = 0$ (which has one complex physical scalar), and y projects to an auxiliary real unconstrained $N = 1$ superfield. This particular off-shell hypermultiplet is discussed extensively in [39]. A superconformal action for O(4) multiplets (which have conformal weight 4) is then constructed from a homogeneous function G of degree 1/2.

For the arctic multiplet, only the two lowest coefficient superfields are constrained. The other components are complex auxiliary superfields unconstrained in $N = 1$ superspace:

$$\Upsilon = \bar{v} + \xi \bar{s} + \xi^2 r_2 + \xi^3 r_3 + \dots \,, \tag{B.14}$$

where v is again constrained as above, and s is a complex linear superfield as in the O(4) case. The arctic multiplet is another off-shell hypermultiplet, in this case with an infinite number of auxiliary fields.

Finally, for the real tropical multiplet all the ξ-coefficient superfields are unconstrained in $N = 1$ superspace.

In general, different multiplets are adapted to different geometries, e.g. O(2) multiplets arise when the hyper-Kähler manifold has commuting triholomorphic isometries. It is believed that polar multiplets can be used to describe arbitrary hyper-Kähler manifolds [37]. The real tropical multiplet is not used to describe sigma-models, but rather arises in the projective superspace description of the $N = 2$ Yang-Mills multiplet [38].

Certain terms in the projective superspace lagrangian G do not contribute to the action (B.5):

$$G_{\rm trivial} = \eta^{(2)} \Big[f_1(\Upsilon) + \bar{f}_1(\bar{\Upsilon}) \Big] + f_2(\Upsilon) + \bar{f}_2(\bar{\Upsilon}) \,, \tag{B.15}$$

where f_i are holomorphic functions. After the contour integral is evaluated, the only terms that survive are either a function of $N = 1$ chiral superfields, or a function of $N = 1$ chiral superfields times an $N = 1$ linear superfield, or the complex conjugates; such terms give rise only to total derivatives.

[13] Superconformal actions in projective and harmonic superspace were also proposed in [33].

B.2 Isometries

The polar multiplet has an infinite number of $N = 1$ superfields; consequently, it is difficult to extract the Kähler potential except in special circumstances. On the other hand, the space of polar multiplets has an algebraic structure: sums and products of arctic multiplets are again arctic, as are holomorphic functions of arctic multiplets. This allows for a very direct realization of triholomorphic isometries of the hyper-Kähler geometry in projective superspace: they are simply symmetries of the projective superspace action (B.5) that are holomorphic in the arctic multiplets. As we explain below, the whole process of gauging triholomorphic isometries and performing hyper-Kähler quotients, when described in terms of polar multiplets in projective superspace is essentially the same procedure as for Kähler quotients described in terms of chiral superfields in $N = 1$ superspace.

We focus on the specific case that applies to the universal hypermultiplet, as this can be trivially generalized to the much larger class of hyper-Kähler metrics that are hyper-Kähler quotients of some (flat) vector space (toric varieties and their non-abelian generalizations). Thus we assume that the isometry is realized linearly on a vector space with coordinates Υ^I (the general case can be treated using the methods of [40]):

$$\delta \Upsilon^I \;=\; i\, \lambda^I{}_J \, \Upsilon^J \,, \tag{B.16}$$

where the parameter λ is a constant matrix in some representation of the isometry group. The invariant projective superspace lagrangian is given by

$$G \;=\; \bar{\Upsilon}^I \eta_{IJ} \Upsilon^J \,. \tag{B.17}$$

For the universal hypermultiplet $I = 1, 2, 3$, and η has signature $- + +$; this is the $N = 2$ projective superspace rewrite of the $N = 1$ superspace lagrangian (7.10) of section 7, as we prove below. This action has a rigid $\mathrm{U}(1, 2)$ group of holomorphic isometries. Any subgroup of this may be gauged by introducing a real tropical gauge multiplet $\mathbf{V}$ that is analogous to the $N = 1$ gauge superfield V, and which transforms as

$$(\mathrm{e}^{\mathbf{V}})'{}_I{}^J = (\mathrm{e}^{i\bar{\lambda}}\, \mathrm{e}^{\mathbf{V}})_I{}^K \, \eta_{KL} \, (\mathrm{e}^{-i\lambda})^L{}_M \, \eta^{MJ} \,, \tag{B.18}$$

where now λ has been generalized to an arctic gauge parameter, and the conjugate gauge parameter $\bar{\lambda}$ is antarctic, as is the conjugate multiplet $\bar{\Upsilon}$. The gauge invariant action is

$$G_{\mathbf{V}} \;=\; \bar{\Upsilon}^I \, (\mathrm{e}^{\mathbf{V}})_I{}^J \, \eta_{JK} \, \Upsilon^K \,. \tag{B.19}$$

One may also add $N = 2$ Fayet-Iliopoulos terms for $\mathrm{U}(1)$ factors in the gauge group; these take the form $c(\xi)\mathbf{V}$ where c is a *constant* $\mathrm{O}(2)$ multiplet, but since these break conformal invariance, they do not interest us here.

These gauged actions and their relation to the discussion of section 7.1 may be understood most directly by going to covariant $N = 1$ superspace components. This is analogous to the vector representation of $N = 1$ Yang-Mills theory: We split the tropical gauge multiplet factors regular at the north and south poles:

$$\mathrm{e}^{\mathbf{V}} = \mathrm{e}^{\mathbf{V}_-}\,\mathrm{e}^{\mathbf{V}_+}\ , \qquad \mathbf{V}_+ = \sum_{n=0}^{\infty} \mathbf{V}_{+n}\,\xi^n\ , \qquad \mathbf{V}_- = \bar{\mathbf{V}}_+ . \tag{B.20}$$

Because $\mathbf{V}$ is an analytic superfield, $\nabla \mathrm{e}^{\mathbf{V}} = 0$, and we may define a gauge-covariant analytic derivative $\boldsymbol{\mathfrak{D}}$

$$\boldsymbol{\mathfrak{D}} \equiv \nabla + \mathrm{e}^{-\mathbf{V}_-}(\nabla \mathrm{e}^{\mathbf{V}_-}) = \nabla - (\nabla \mathrm{e}^{\mathbf{V}_+})\mathrm{e}^{-\mathbf{V}_+} . \tag{B.21}$$

Comparing powers of ξ for both expressions, we conclude that $\boldsymbol{\mathfrak{D}}$ has only a constant and a linear term (just as ∇), and hence defines the $N = 2$ gauge-covariant derivative (for a more detailed explanation see [38]). In particular, we find the covariantly chiral gauge field strength $\mathbf{W}$ by computing

$$\{\bar{\boldsymbol{\mathfrak{D}}}_{\dot\alpha}(\xi_1), \bar{\boldsymbol{\mathfrak{D}}}_{\dot\beta}(\xi_2)\} \;=\; \varepsilon_{\dot\alpha\dot\beta}\,(\xi_2 - \xi_1)\,\mathbf{W} . \tag{B.22}$$

Note that $\mathbf{W}$ is ξ independent. We may also define *covariantly* analytic polar multiplets

$$\hat\Upsilon \equiv \mathrm{e}^{\mathbf{V}_+}\Upsilon\ , \qquad \hat{\bar\Upsilon} = \bar\Upsilon \mathrm{e}^{\mathbf{V}_-} . \tag{B.23}$$

In terms of these, the gauge-invariant lagrangian (B.19) is just quadratic, the ξ integral (B.5) can be trivially evaluated, and the auxiliary superfields can be integrated out to get the gauge-invariant $N = 1$ superspace lagrangian

$$L_{N=1} = \hat{\bar v}^I \hat v^J \eta_{IJ} - \hat{\bar s}^I \hat s^J \eta_{IJ}\ , \tag{B.24}$$

where $\hat v^I$ are $N = 1$ gauge-covariantly (vector representation) chiral superfields and $\hat s^I$ are modified $N = 1$ gauge-covariantly complex linear superfields

$$\bar{\mathcal{D}}_{\dot\alpha} \hat v^I = 0\ , \qquad \bar{\mathcal{D}}^2 \hat s^I = \hat W^I{}_J \hat v^J . \tag{B.25}$$

Here $\hat W^I{}_J$ is the $N = 1$ covariantly chiral projection of the $N = 2$ field strength $\mathbf{W}$ (B.22) in the representation that acts on $\hat v^J$ and $\mathcal{D}$ is the $N = 1$ gauge-covariant derivative. We can go to chiral representation and replace $\hat v, \hat s, \hat W$ with ordinary chiral and linear superfields v, s, W by introducing the $N = 1$ gauge potential V:

$$(\mathrm{e}^V)_I{}^K\,\eta_{KJ} \equiv (\mathrm{e}^{V_-})_I{}^K\,\eta_{KL}(\mathrm{e}^{V_+})^L{}_J\ , \quad \hat v = \mathrm{e}^{V_+} v\ , \quad \hat s = \mathrm{e}^{V_+} s\ , \quad \hat W = \mathrm{e}^{V_+} W \mathrm{e}^{-V_+}\ , \tag{B.26}$$

where $V_\pm$ is the $N = 1$ projection of the ξ-independent coefficients of $\mathbf{V}_\pm$. These substitutions lead to

$$L_{N=1} = (\bar v\,\mathrm{e}^V)^I \eta_{IJ} v^J - (\bar s\,\mathrm{e}^V)^I \eta_{IJ} s^J\ , \tag{B.27}$$

$$\bar D^2 s^I = W^I{}_J v^J . \tag{B.28}$$

It is convenient to rewrite the $N = 1$ lagrangian (B.27) in terms of chiral superfields; to do this, we impose the constraints (B.28) by chiral Lagrange multipliers z_{-I} in a superpotential term

$$z_{-I}(\bar{D}^2 s^I - W^I{}_J v^J)\,, \tag{B.29}$$

and integrate out s to obtain the non-abelian generalization of the $N = 1$ gauged lagrangian (7.17) (after relabeling $v \to z_+$):

$$L_{N=1} = (\bar{z}_+ \mathrm{e}^V)^I \eta_{IJ} z_+{}^J + z_{-I}\eta^{IJ}(\mathrm{e}^{-V}\bar{z}_-)_J\,. \tag{B.30}$$

In addition, we are left with a superpotential term

$$\mathrm{Tr}\,[\,W\mu^+] = z_{-I} W^I{}_J z_+^J\,, \tag{B.31}$$

where μ^+ is just (the non-abelian generalization of) the holomorphic moment map (7.12). Observe that interchanging $z_+ \leftrightarrow z_-$ and changing the representation of V to its conjugate does not modify the gauged lagrangian (B.30); this implies that in the original $N = 2$ lagrangian $G_{\mathbf{V}}$ (B.19), we can take Υ transforming in the conjugate representation (e.g. opposite charge for U(1)) without changing the final result. In the next subsection we integrate out the $N = 2$ gauge fields to find the quotient lagrangian; in $N = 1$ superspace, integrating out the chiral superfield W imposes the moment map constraint $\mu^+ = 0$.

B.3 Quotients and duality

Just as $N = 2$ isometries and gauging in projective superspace bear a striking resemblance to their $N = 1$ superspace analogs, so do $N = 2$ quotients and duality; indeed, the tensor multiplet projective superspace lagrangian is just the Legendre transform of the polar multiplet lagrangian.

The procedure we follow is the same as in $N = 1$ superspace: we gauge the relevant isometries as above; to perform a quotient, we simply integrate out the gauge prepotential $\mathrm{e}^{\mathbf{V}}$. Since this does not break the isometry, we are left with an action defined on the quotient space. To find the dual, we add a Lagrange multiplier η that constrains the gauge prepotential to be trivial,[14] and again integrate out $\mathbf{V}$; the dual field is then the Lagrange multiplier η. As in the $N = 1$ case, we only consider duality for abelian isometries. In that case, the Lagrange multiplier term that constrains $\mathbf{V}$ is

$$\eta\mathbf{V}\,, \tag{B.32}$$

where η is the O(2) superfield that describes the $N = 2$ tensor multiplet as explained above (B.12).

[14] As explained in [41, 42], this is the correct geometric way of understanding duality; when one chooses coordinates such that the Killing vectors generating the isometries are constant, this gives the usual Legendre transform.

JHEP02(2001)039

We have now assembled all the tools we need to find the various tensor multiplet formulations of the universal hypermultiplet.

The HKC of the universal hypermultiplet is just the U(1) hyper-Kähler quotient of a flat lorentzian space:

$$\hat{G}_{\mathbf{V}} = (-\bar{\Upsilon}^1\Upsilon^1 + \bar{\Upsilon}^2\Upsilon^2 + \bar{\Upsilon}^3\Upsilon^3)\,\mathrm{e}^{\mathbf{V}}\,. \tag{B.33}$$

This is manifestly invariant under $\mathrm{U}(1,2)$; however, if we integrate out $\mathbf{V}$, we get a constraint (the moment map constraints in projective superspace) rather than an $N=2$ superspace lagrangian — even though we do get a corresponding $N=1$ chiral superfield quotient lagrangian from (B.30). We can get an $N=2$ quotient lagrangian at the expense of losing manifest $\mathrm{U}(1,2)$ invariance by using the observation from the end of the previous subsection that the final $N=1$ superspace result does not change when we take $\mathbf{V}\to-\mathbf{V}$:

$$\check{G}_{\mathbf{V}} = (-\bar{\Upsilon}^1\Upsilon^1 + \bar{\Upsilon}^2\Upsilon^2)\,\mathrm{e}^{\mathbf{V}} + \bar{\Upsilon}^3\Upsilon^3\,\mathrm{e}^{-\mathbf{V}}\,. \tag{B.34}$$

Integrating out $\mathbf{V}$ and fixing the gauge $\Upsilon^3=1$ gives the polar multiplet superspace lagrangian for the hyper-Kähler cone of the universal hypermultiplet:

$$G_{\mathrm{HKC}} = 2\sqrt{-\bar{\Upsilon}^1\Upsilon^1 + \bar{\Upsilon}^2\Upsilon^2}\,, \tag{B.35}$$

which has manifest $\mathrm{U}(1,1)$ rather than $\mathrm{SU}(1,2)$ invariance.

We now dualize (B.33) with respect to any pair of commuting holomorphic isometries as discussed above by gauging the isometries and adding a term $\eta^1\mathbf{V}_1+\eta^2\mathbf{V}_2$. The three choices of inequivalent commuting pairs of U(1) generators are given in (7.44)– (7.46). Exponentiating the generators gives three different expressions:

$$\mathrm{e}^{\mathbf{V}_1T_1+\mathbf{V}_2T_2} = \begin{pmatrix} \mathrm{e}^{-\mathbf{V}_1-\mathbf{V}_2} & 0 & 0 \\ 0 & \mathrm{e}^{-\mathbf{V}_1+\mathbf{V}_2} & 0 \\ 0 & 0 & \mathrm{e}^{2\mathbf{V}_1} \end{pmatrix}, \tag{B.36}$$

$$\mathrm{e}^{\mathbf{V}_1T_1+\mathbf{V}_2T_2'} = \begin{pmatrix} \mathrm{e}^{-\mathbf{V}_1} & 0 & 0 \\ 0 & \mathrm{e}^{-\mathbf{V}_1} & 0 \\ 0 & 0 & \mathrm{e}^{2\mathbf{V}_1} \end{pmatrix}\begin{pmatrix} 1+\mathbf{V}_2 & \mathbf{V}_2 & 0 \\ -\mathbf{V}_2 & 1-\mathbf{V}_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{B.37}$$

$$\begin{aligned}\mathrm{e}^{\mathbf{V}_1T_1'+\mathbf{V}_2T_2'} &= \begin{pmatrix} 1-1/2\mathbf{V}_1^2 & -1/2\mathbf{V}_1^2 & -\mathbf{V}_1 \\ 1/2\mathbf{V}_1^2 & 1+1/2\mathbf{V}_1^2 & \mathbf{V}_1 \\ \mathbf{V}_1 & \mathbf{V}_1 & 1 \end{pmatrix}\begin{pmatrix} 1+\mathbf{V}_2 & \mathbf{V}_2 & 0 \\ -\mathbf{V}_2 & 1-\mathbf{V}_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1+\mathbf{V}_2-1/2\mathbf{V}_1^2 & \mathbf{V}_2-1/2\mathbf{V}_1^2 & -\mathbf{V}_1 \\ -\mathbf{V}_2+1/2\mathbf{V}_1^2 & 1-\mathbf{V}_2+1/2\mathbf{V}_1^2 & \mathbf{V}_1 \\ \mathbf{V}_1 & \mathbf{V}_1 & 1 \end{pmatrix}.\end{aligned} \tag{B.38}$$

It is now straightforward to evaluate the gauged action $\hat{G}_{\mathbf{V}}$ (B.33), add $\eta^1\mathbf{V}_1+\eta^2\mathbf{V}_2$, and integrate out $\mathbf{V}$, $\mathbf{V}_1$ and $\mathbf{V}_2$. Up to signs that can be fixed by choosing the orientations of the contours, trivial terms (B.15), and linear redefinitions of the η^I, we find the three dual lagrangians (7.43), (7.47), and (7.48).

One may also dualize the U$(1,1)$ invariant lagrangian G_{HKC} (B.35). Here there are only two choices of inequivalent pairs of commuting generators: the 2×2 identity matrix $\mathbf{1}_2$ and one of

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} . \tag{B.39}$$

Again up to signs having to do with the orientations of the contours, as well as irrelevant terms, these give only the two lagrangians (7.43) and (7.47).

It is also instructive to consider the example of anti-de-Sitter space AdS_4. The HKC is eight-dimensional flat space with a polar multiplet lagrangian

$$G_{AdS_4} = -\bar{\Upsilon}^1\Upsilon^1 + \bar{\Upsilon}^2\Upsilon^2 , \tag{B.40}$$

that is U$(1,1)$ invariant. This can be dualized with respect to the same inequivalent pairs of commuting generators as in the previous paragraph; the resulting tensor actions are

$$G^1_{AdS_4} = -\eta^1 \ln \eta^1 + \eta^2 \ln \eta^2 , \qquad G^2_{AdS_4} = -\eta^1 \ln \eta^2 , \tag{B.41}$$

respectively.

References

[1] M. Atiyah and N. Hitchin, *The Geometry and Dynamics of Magnetic Monopoles*, Princeton University Press, Princeton 1988.

[2] M.F. Atiyah, N.J. Hitchin, V.G. Drinfeld and Y.I. Manin, *Construction of instantons*, *Phys. Lett.* **A 65** (1978) 185.

[3] L. Alvarez-Gaume and D.Z. Freedman, *Geometrical structure and ultraviolet finiteness in the supersymmetric sigma model*, *Comm. Math. Phys.* **80** (1981) 443.

[4] J.D. Jaegher, B. de Wit, B. Kleijn and S. Vandoren, *Special geometry in hypermultiplets*, *Nucl. Phys.* **B 514** (1998) 553 [hep-th/9707262].

[5] J. Bagger and E. Witten, *Matter couplings in $N = 2$ supergravity*, *Nucl. Phys.* **B 222** (1983) 1.

[6] S. Cecotti, S. Ferrara and L. Girardello, *Geometry of type II superstrings and the moduli of superconformal field theories*, *Int. J. Mod. Phys.* **A 4** (1989) 2475;
S. Ferrara and S. Sabharwal, *Quaternionic manifolds for type II superstring vacua of Calabi-Yau spaces*, *Nucl. Phys.* **B 332** (1990) 317.

[7] A. Strominger, *Loop corrections to the universal hypermultiplet*, *Phys. Lett.* **B 421** (1998) 139 [hep-th/9706195];
E. Kiritsis and C. Kounnas, *Infrared regularization of superstring theory and the one loop calculation of coupling constants*, *Nucl. Phys.* **B 442** (1995) 472 [hep-th/9501020];
I. Antoniadis, S. Ferrara, R. Minasian and K.S. Narain, *R^4 couplings in M- and type II theories on Calabi-Yau spaces*, *Nucl. Phys.* **B 507** (1997) 571 [hep-th/9707013];
H. Gunther, C. Herrmann and J. Louis, *Quantum corrections in the hypermultiplet moduli space*, *Fortsch. Phys.* **48** (2000) 119 [hep-th/9901137].

[8] K. Becker, M. Becker and A. Strominger, *Five-branes, membranes and nonperturbative string theory*, *Nucl. Phys.* **B 456** (1995) 130 [hep-th/9507158];
H. Ooguri and C. Vafa, *Summing up D instantons*, *Phys. Rev. Lett.* **77** (1996) 3296, [hep-th/9608079] K. Becker and M. Becker, *Instanton action for type II hypermultiplets*, *Nucl. Phys.* **B 551** (1999) 102 [hep-th/9901126];
M. Gutperle and M. Spalinski, *Supergravity instantons and the universal hypermultiplet*, *J. High Energy Phys.* **06** (2000) 037 [hep-th/0005068];
M. Gutperle and M. Spalinski, *Supergravity instantons for $N = 2$ hypermultiplets*, hep-th/0010192;
S.V. Ketov, *Gravitational dressing of d-instantons*, hep-th/0010255.

[9] B. de Wit, J.W. van Holten and A.V. Proeyen, *Transformation rules of $N = 2$ supergravity multiplets*, *Nucl. Phys.* **B 167** (1980) 186;
B. de Wit, P.G. Lauwers and A.V. Proeyen, *Lagrangians of $N = 2$ supergravity - matter systems*, *Nucl. Phys.* **B 255** (1985) 569.

[10] B. de Wit, B. Kleijn and S. Vandoren, *Rigid $N = 2$ superconformal hypermultiplets*, in *Supersymmetries and Quantum Symmetries*, Lecture Notes in Physics, vol. 524, Springer 1999 hep-th/9808160.

[11] B. de Wit, B. Kleijn and S. Vandoren, *Superconformal hypermultiplets*, *Nucl. Phys.* **B 568** (2000) 475 [hep-th/9909228].

[12] A. Swann, *HyperKähler and quaternionic Kähler geometry*, *Math. Ann.*, **289** (1991) 421.

[13] K. Galicki, *Geometry of the scalar couplings in $N = 2$ supergravity models*, *Class. and Quant. Grav.* **9** (1992) 27.

[14] P.S. Howe, A. Opfermann and G. Papadopoulos, *Twistor spaces for QKT manifolds*, *Comm. Math. Phys.* **197** (1998) 713 [hep-th/9710072].

[15] S.M. Salamon, *Quaternionic Kähler manifolds*, *Invent. Math.*, **67** (1982) 143.

[16] A. Karlhede, U. Lindstrom and M. Rocek, *Selfinteracting tensor multiplets in $N = 2$ superspace*, *Phys. Lett.* **B 147** (1984) 297.

JHEP02(2001)039

[17] N.J. Hitchin, A. Karlhede, U. Lindstrom and M. Rocek, *Hyperkahler metrics and supersymmetry*, *Comm. Math. Phys.* **108** (1987) 535.

[18] B. de Wit, M. Roček and S. Vandoren, in preparation.

[19] G.W. Gibbons and P. Rychenkova, *Cones, tri-sasakian structures and superconformal invariance*, *Phys. Lett.* **B 443** (1998) 138 [hep-th/9809158].

[20] C.P. Boyer and K. Galicki, *3-Sasakian manifolds*, hep-th/9810250.

[21] J. Marsden and A. Weinstein, *Reduction of symplectic manifolds with symmetry*, *Rep. Math. Phys.*, **5** (1974) 121

[22] U. Lindström and M. Roček, *Scalar tensor duality and $N = 1$, $N = 2$ nonlinear sigma models*, *Nucl. Phys.* **B 222** (1983) 285

[23] B. de Wit, F. Vanderseypen and A.V. Proeyen, *Symmetry structure of special geometries*, *Nucl. Phys.* **B 400** (1993) 463 [hep-th/9210068].

[24] B. de Wit, R. Philippe and A.V. Proeyen, *The improved tensor multiplet in $N = 2$ supergravity*, *Nucl. Phys.* **B 219** (1983) 143.

[25] J. Wolf, *Complex homogeneous contact manifolds and quaternionic symmetric spaces*, *J. Math. Mech.*, **14** (1965) 1033

[26] R. Gilmore, *Lie Groups, Lie Algebras and Some of Their Applications*, Wiley eds 1974.

[27] P. Kobak and A. Swann, *Hyper-Kähler potentials in cohomogeneity two*, math.DG/0001024.

[28] P. Breitenlohner and M.F. Sohnius, *Matter coupling and nonlinear sigma models in $N = 2$ supergravity*, *Nucl. Phys.* **B 187** (1981) 409.

[29] K. Galicki, *Quaternionic kahler and hyperkahler nonlinear sigma models*, *Nucl. Phys.* **B 271** (1986) 402;
A generalization of the momentum mapping construction for quaternionic kahler manifolds, *Comm. Math. Phys.* **108** (1987) 117.

[30] N. Berkovits and W. Siegel, *Superspace effective actions for 4D compactifications of heterotic and type II superstrings*, *Nucl. Phys.* **B 462** (1996) 213 [hep-th/9510106];
W. Siegel, *Curved extended superspace from Yang-Mills theory a la strings*, *Phys. Rev.* **D 53** (1996) 3324 [hep-th/9510150].

[31] S. Vandoren, *Instantons and quaternions*, hep-th/0009150.

[32] A. Maciocia, *Metrics on the moduli spaces of instantons over euclidean four space*, *Comm. Math. Phys.* **135** (1991) 467.

[33] S.V. Ketov, *Superconformal hypermultiplets in superspace*, *Nucl. Phys.* **B 582** (2000) 95 [hep-th/0001109];
E. Ivanov and G. Valent, *Quaternionic metrics from harmonic superspace: lagrangian approach and quotient construction*, *Nucl. Phys.* **B 576** (2000) 543 [hep-th/0001165].

[34] N. Seiberg and S. Shenker, *Hypermultiplet moduli space and string compactification to three dimensions*, *Phys. Lett.* **B 388** (1996) 521 [hep-th/9608086];
E. Witten, *Heterotic string conformal field theory and a-d-e singularities*, *J. High Energy Phys.* **02** (2000) 025 [hep-th/9909229];
M. Rozali, *Hypermultiplet moduli space and three dimensional gauge theories*, *J. High Energy Phys.* **12** (1999) 013 [hep-th/9910238].

[35] N. Seiberg and E. Witten, *Gauge dynamics and compactification to three dimensions*, hep-th/9607163;
N. Dorey, V.V. Khoze, M.P. Mattis, D. Tong and S. Vandoren, *Instantons, three-dimensional gauge theory and the Atiyah- Hitchin manifold*, *Nucl. Phys.* **B 502** (1997) 59 [hep-th/9703228].

[36] F. Gonzalez-Rey, M. Rocek, S. Wiles, U. Lindstrom and R. von Unge, *Feynman rules in $N = 2$ projective superspace I: massless hypermultiplets*, *Nucl. Phys.* **B 516** (1998) 426 [hep-th/9710250].

[37] U. Lindström and M. Roček, *New hyperkahler metrics and new supermultiplets*, *Comm. Math. Phys.* **115** (1988) 21.

[38] U. Lindström, M. Roček, *$N = 2$ SuperYang-Mills theory in projective superspace*, *Comm. Math. Phys.* **128** (1990) 191.

[39] I.T. Ivanov and M. Rocek, *Supersymmetric sigma models, twistors and the Atiyah-Hitchin metric*, *Comm. Math. Phys.* **182** (1996) 291 [hep-th/9512075].

[40] C.M. Hull, A. Karlhede, U. Lindstrom and M. Rocek, *Nonlinear sigma models and their gauging in and out of superspace*, *Nucl. Phys.* **B 266** (1986) 1.

[41] P. Howe, A. Karlhede, U. Lindström and M. Roček, *The geometry of duality*, *Phys. Lett.* **B 168** (1986) 89.

[42] M. Rocek and E. Verlinde, *Duality, quotients and currents*, *Nucl. Phys.* **B 373** (1992) 630 [hep-th/9110053].